Existence of a ground state and scattering for a nonlinear Schrödinger equation with critical growth

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Abstract

We study the energy-critical focusing nonlinear Schrödinger equation with an energy-subcritical perturbation. We show the existence of a ground state in the four or higher dimensions. Moreover, we give a sufficient and necessary condition for a solution to scatter, in the spirit of Kenig-Merle [16].

1 Introduction

In this paper, we study the following nonlinear Schrödinger equation.

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + f(\psi) + |\psi|^{2^*-2}\psi = 0, \tag{NLS}$$

where $\psi = \psi(x,t)$ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$ ($d \geq 3$), Δ is the Laplace operator on $\mathbb{R}^d$, $2^* := 2 + \frac{4}{d-2}$ and $f\colon \mathbb{C} \to \mathbb{C}$ is a continuously differentiable function in the $\mathbb{R}^2$-sense. We specify the nonlinearity f later (see the assumptions (A0)–(A6) below); Especially, we assume the Hamiltonian structure (see (A1) below), so that there exists a function $F \in C^2(\mathbb{C}, \mathbb{R})$ such that the Hamiltonian for (NLS) is given by

$$\mathcal{H}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}^d} F(u) - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}. \tag{1.1}$$

Moreover, we assume that f satisfies the mass-supercritical and energy-subcritical condition (see (A5) and (A6)). Hence, the equation (NLS) is considered to be a perturbed one of

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + |\psi|^{2^*-2}\psi = 0. \tag{NLS$_0$}$$

Here, the Hamiltonian for (NLS$_0$) is

$$\mathcal{H}_0(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}. \tag{1.2}$$

It is well-known that the equation (NLS$_0$) dose not have an oscillatory standing wave; In contrast, (NLS$_0$) has the non-oscillatory solution

$$W(x) := \left(\frac{\sqrt{d(d-2)}}{1+|x|^2}\right)^{\frac{d-2}{d}}, \qquad x \in \mathbb{R}^d. \tag{1.3}$$

Our first aim is to show that a suitable perturbation f gives rise to an oscillatory standing wave for $d \geq 4$. In other words, we intend to prove that for $d \geq 4$ and $\omega > 0$, there exists a solution to the elliptic equation

$$-\Delta u + \omega u - f(u) - |u|^{2^*-2}u = 0, \qquad u \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{1.4}$$

In particular, we show the existence of a ground state for $d \geq 4$ (see Theorem 1.1 and Remark 1.1 below); A ground state means a solution to (1.4) which minimizes the action $\mathcal{S}_\omega$ among the solutions, where

$$\mathcal{S}_\omega(u) := \frac{\omega}{2} \|u\|_{L^2}^2 + \mathcal{H}(u). \tag{1.5}$$

We remark that, in [1], the same authors considered the case $f(z) = \mu|z|^{p-1}z$ with $\mu \in \mathbb{R}$ and $2_* - 1 < p < 2^* - 1$ ($2_* := 2 + \frac{4}{d}$), and proved that if $d \geq 4$, then for any $\mu > 0$ and $\omega > 0$, there exists a ground state; on the other hand, if $d \geq 3$ and $\mu \leq 0$, or $d = 3$ and $\mu/\omega^{\frac{d-2}{4}\{2^*-(p+1)\}}$ is sufficiently small, then the equation (1.4) has no solution.

In this paper, we extend the result in [1] to a wider class of perturbations including

$$f(z) = \mu_1|z|^{p_1-1}z + \cdots + \mu_k|z|^{p_k-1}z, \tag{1.6}$$

where $k \in \mathbb{N}$, $\mu_1, \ldots, \mu_k > 0$ and $2_* - 1 < p_1 < \cdots < p_k < 2^* - 1$.

Our second aim is to give a necessary and sufficient condition for solutions to scatter, in the spirit of Kenig-Merle [16], for $d \geq 5$ (see Theorem 1.2 below); Precisely, we introduce a set $A_{\omega,+}$ (see (1.18) below) and prove that any solution starting from $A_{\omega,+}$ exists globally in time and asymptotically behaves like a free solution in the distant future and past. Although we can introduce the set $A_{\omega,+}$ for $d \geq 3$, the scattering result is open in $d = 3, 4$, as well as the equation (NLS_0) (see [18]).

Now, we state our assumption of the perturbation f. We first assume that

$$f(0) = \frac{\partial f}{\partial z}(0) = \frac{\partial f}{\partial \bar{z}}(0) = 0. \tag{A0}$$

For the Hamiltonian structure and mass conservation law, we assume that there exists a real-valued function $F \in C^2(\mathbb{C}, \mathbb{R})$ such that

$$F(0) = 0, \qquad \frac{\partial F}{\partial \bar{z}} = f, \qquad \Im\left[z\frac{\partial F}{\partial z} \right] = 0, \tag{A1}$$

so that $\bar{z}f(z) \in \mathbb{R}$ for any $z \in \mathbb{C}$. Besides, we assume that

$$F \geq 0. \tag{A2}$$

This assumption (A2) rules out the case $F(z) = -\frac{1}{p+1}|z|^{p+1}$ (or $f(z) = -|z|^{p-1}z$) for which the Pohozaev identity shows that there is no solution to (1.4).

To ensure the existence of ground state, we further need the monotonicity and convexity conditions, like in [13]: Define the operator D by

$$Dg(z) := z\frac{\partial g}{\partial z}(z) + \bar{z}\frac{\partial g}{\partial \bar{z}}(z) \qquad \text{for } g \in C^1(\mathbb{C}, \mathbb{C}). \tag{1.7}$$

Then, we assume that there exists $\varepsilon_0 > 0$ such that

$$(D - 2_* - \varepsilon_0)F \geq 0, \tag{A3}$$

$$(D - 2)(D - 2_* - \varepsilon_0)F \geq 0. \tag{A4}$$

The conditions (A2), (A3) and (A4) imply that

$$D^2F \geq (2_* + \varepsilon_0)DF \geq (2_* + \varepsilon_0)^2F \geq 0. \tag{1.8}$$

Finally, we make an assumption so that f satisfies the mass-supercritical and energy-subcritical growth, like in [13]: Fix a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(u) = 1$ for $|u| \le 1$ and $\chi(u) = 0$ for $|u| \ge 2$, and put $f_{\le 1}(u) := \chi(u) f(u)$ and $f_{\ge 1} := f - f_{\le 1}$. Then, we assume that there exist p_1 and p_2 such that $2_* - 1 < p_1 \le p_2 < 2^* - 1$, and

$$\begin{cases} \left|\frac{\partial f_{\le 1}}{\partial z}(u)\right| + \left|\frac{\partial f_{\le 1}}{\partial \bar{z}}(u)\right| \lesssim |u|^{p_1 - 1} & \text{if } d = 3, 4, \\ \left|\frac{\partial f_{\le 1}}{\partial z}(u) - \frac{\partial f_{\le 1}}{\partial z}(v)\right| + \left|\frac{\partial f_{\le 1}}{\partial \bar{z}}(u) - \frac{\partial f_{\le 1}}{\partial \bar{z}}(v)\right| \lesssim (|u| + |v|)^{p_1 - 2}|u - v| & \text{if } d \ge 5,\ p_1 \ge 2, \\ \left|\frac{\partial f_{\le 1}}{\partial z}(u) - \frac{\partial f_{\le 1}}{\partial z}(v)\right| + \left|\frac{\partial f_{\le 1}}{\partial \bar{z}}(u) - \frac{\partial f_{\le 1}}{\partial \bar{z}}(v)\right| \lesssim |u - v|^{p_1 - 1} & \text{if } d \ge 5,\ p_1 < 2 \end{cases} \tag{A5}$$

and

$$\begin{cases} \left|\frac{\partial f_{\ge 1}}{\partial z}(u)\right| + \left|\frac{\partial f_{\ge 1}}{\partial \bar{z}}(u)\right| \lesssim |u|^{p_2 - 1} & \text{if } d = 3, 4, \\ \left|\frac{\partial f_{\ge 1}}{\partial z}(u) - \frac{\partial f_{\ge 1}}{\partial z}(v)\right| + \left|\frac{\partial f_{\ge 1}}{\partial \bar{z}}(u) - \frac{\partial f_{\ge 1}}{\partial \bar{z}}(v)\right| \lesssim (|u| + |v|)^{p_2 - 2}|u - v| & \text{if } d \ge 5,\ p_2 \ge 2, \\ \left|\frac{\partial f_{\ge 1}}{\partial z}(u) - \frac{\partial f_{\ge 1}}{\partial z}(v)\right| + \left|\frac{\partial f_{\ge 1}}{\partial \bar{z}}(u) - \frac{\partial f_{\ge 1}}{\partial \bar{z}}(v)\right| \lesssim |u - v|^{p_2 - 1} & \text{if } d \ge 5,\ p_2 < 2. \end{cases} \tag{A6}$$

As mentioned above, we prove the existence of a ground state to the equation (1.4) under the assumptions (A0)–(A6). To this end, for any $\omega > 0$, we introduce a variational value m_ω:

$$m_\omega := \inf \left\{ \mathcal{S}_\omega(u) \;\middle|\; u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{K}(u) = 0 \right\}, \tag{1.9}$$

where

$$\begin{aligned} \mathcal{K}(u) &:= \frac{d}{d\lambda} \mathcal{H}(T_\lambda u) \bigg|_{\lambda=1} = \frac{d}{d\lambda} \mathcal{S}_\omega(T_\lambda u) \bigg|_{\lambda=1} \\ &= \|\nabla u\|_{L^2}^2 - \frac{d}{4} \int_{\mathbb{R}^d} \left(DF - 2F \right)(u) - \|u\|_{L^{2^*}}^{2^*} \end{aligned} \tag{1.10}$$

and T_λ is the L^2-scaling operator, i.e.,

$$(T_\lambda u)(x) := \lambda^{\frac{d}{2}} u(\lambda x). \tag{1.11}$$

It is well-known(see, e.g., [4, 19]) that the minimizer of (1.9) becomes a ground state to (1.4). Thus, it suffices to show the existence of the minimizer.

In order to find the minimizer of the variational problem (1.9), we need two auxiliary variational problems; The first one is

$$\widetilde{m}_\omega = \inf \left\{ \mathcal{I}_\omega(u) \;\middle|\; u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{K}(u) \le 0 \right\}, \tag{1.12}$$

where

$$\begin{aligned} \mathcal{I}_\omega(u) &:= \mathcal{S}_\omega(u) - \frac{1}{2} \mathcal{K}(u) \\ &= \frac{\omega}{2} \|u\|_{L^2}^2 + \frac{d}{8} \int_{\mathbb{R}^d} \left(DF - 2_* F \right)(u) + \frac{1}{d} \|u\|_{L^{2^*}}^{2^*}, \end{aligned} \tag{1.13}$$

and the other one is

$$\sigma := \inf \left\{ \|\nabla u\|_{L^2}^2 \;\middle|\; u \in \dot{H}^1(\mathbb{R}^d),\ \|u\|_{L^{2^*}} = 1 \right\}. \tag{1.14}$$

An advantage of the problem (1.12) is that the functional $\mathcal{I}_\omega$ is positive thanks to (A3), and the constraint $\mathcal{K} \le 0$ is stable under the Schwarz symmetrization. Moreover, we have;

Proposition 1.1. *Assume $d \ge 3$, $\omega > 0$ and the conditions* (A0)–(A6)*. Then, it holds that*

(i) $m_\omega = \widetilde{m}_\omega$,

(ii) *Any minimizer of the variational problem for $\widetilde{m}_\omega$ is also a minimizer for m_ω.*

The reason why we need the variational problem (1.14) is the following relation between m_ω and σ:

Lemma 1.2. *Assume $d \ge 4$, $\omega > 0$ and the conditions* (A0)–(A6)*. Then, we have*

$$m_\omega < \frac{1}{d}\sigma^{\frac{d}{2}}. \tag{1.15}$$

Here, it is worthwhile noting that the function W given in (1.3) relates to the value σ:

$$\sigma^{\frac{d}{2}} = \left\|T'_\lambda \nabla W\right\|_{L^2}^2 = \left\|T'_\lambda W\right\|_{L^{2^*}}^{2^*} \quad \text{for any } \lambda > 0, \tag{1.16}$$

where T'_λ denotes the $\dot{H}^1$-scaling operator, i.e.,

$$(T'_\lambda u)(x) := \lambda^{\frac{d-2}{2}} u(\lambda x). \tag{1.17}$$

Since the proof of Lemma 1.2 is similar to the one of Lemma 2.2 in [1], we omit it.

Using Lemma 1.2, we can find:

Theorem 1.1. *Assume $d \ge 4$, $\omega > 0$ and the conditions* (A0)–(A6)*. Then, there exists a minimizer of the variational problem for m_ω.*

Remark 1.1. (i) *We find from Proposition 1.1 (i) that $m_\omega = \widetilde{m}_\omega > 0$.*

(ii) *It is necessary that $d \ge 4$; For, when $f(u) = \mu|u|^{p_1-1}u$ with a sufficiently small $\mu > 0$, there is no minimizer of the variational problem for m_ω (see [1]).*

(iii) *As mentioned above, a minimizer of the variational problem for m_ω becomes a ground state to* (1.4) *(see [4, 19]).*

In order to state our scattering result, we introduce a set $A_{\omega,+}$:

$$A_{\omega,+} := \left\{ u \in H^1(\mathbb{R}^d) \;\middle|\; \mathcal{S}_\omega(u) < m_\omega,\ \mathcal{K}(u) > 0. \right\}. \tag{1.18}$$

Then, we have:

Theorem 1.2. *Assume $d \ge 5$, $\omega > 0$ and the conditions* (A0)–(A6)*. Then, any solution ψ to* (NLS) *starting from $A_{\omega,+}$ exists globally in time. Furthermore, the solution ψ scatters in $H^1(\mathbb{R}^d)$, i.e., there exist $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that*

$$\lim_{t\to+\infty} \left\|\psi(t) - e^{it\Delta}\phi_+\right\|_{H^1} = \lim_{t\to-\infty} \left\|\psi(t) - e^{it\Delta}\phi_-\right\|_{H^1} = 0. \tag{1.19}$$

This paper is organised as follows. In Section 2, we summarise basic properties of the functionals $\mathcal{S}_\omega$, $\mathcal{K}$ and so on. In Section 3, we give proofs of Proposition 1.1 and Theorem 1.1. In Section 4, we introduce Strichartz type spaces and discuss the well-posedness for the equation (NLS). In Section 5, we give a long-time perturbation theory which plays an important role to prove the scattering result(Theorem 1.2). In Section 6, we prove Theorem 1.2 by showing the existence of the so-called critical element in a reductive absurdity.

Finally, we give several notation used in this paper:

Notation.

1. $$2_* := 2 + \frac{4}{d}, \qquad 2^* := 2 + \frac{4}{d-2} \tag{1.20}$$

2. $$s_p := \frac{d}{2} - \frac{2}{p-1} \quad \text{for } p > 1. \tag{1.21}$$

3. We denote the Hölder conjugate of $q \in (1, \infty)$ by q', i.e, $q' = \frac{q}{q-1}$.

4. Let A and B be two positive quantities. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$, where C can depend on d, ε_0 in (A3), p_1 in (A5), p_2 in (A6) and a given ω.

2 Preliminaries

In this section, we give basic properties of functionals $\mathcal{S}_\omega$, $\mathcal{K}$ and so on.

We first summarize easy fact of calculation, without the proofs:

$$\frac{d}{d\lambda}\mathcal{S}_\omega(T_\lambda u) = \frac{d}{d\lambda}\mathcal{H}(T_\lambda u) = \frac{1}{\lambda}\mathcal{K}(T_\lambda u), \tag{2.1}$$

$$\frac{d}{d\lambda}\int_{\mathbb{R}^d}\big(DF - \alpha F\big)(T_\lambda u) = \frac{d}{2\lambda}\int_{\mathbb{R}^d}\big(D^2F - (2+\alpha)DF + 2\alpha F\big)(T_\lambda u) \quad \text{for any } \alpha \in \mathbb{R}. \tag{2.2}$$

We also see from (2.2) together with (A3) and (A4) that

$$\frac{d}{d\lambda}\int_{\mathbb{R}^d}\big(\, DF - 2F\,\big)\,(T_\lambda u) \ge 0, \tag{2.3}$$

$$\frac{d}{d\lambda}\int_{\mathbb{R}^d}\big(\, DF - 2_*F\,\big)\,(T_\lambda u) \ge 0, \tag{2.4}$$

$$\frac{d}{d\lambda}\left\{\frac{1}{\lambda}\int_{\mathbb{R}^d}\big(\, DF - 2F\,\big)\,(T_\lambda u)\right\} \ge 0, \tag{2.5}$$

$$\frac{d}{d\lambda}\left\{\frac{1}{\lambda^2}\int_{\mathbb{R}^d}\big(\, DF - 2F\,\big)\,(T_\lambda u)\right\} \ge 0. \tag{2.6}$$

Next, we give important properties of the functionals $\mathcal{K}$ and $\mathcal{I}_\omega$:

Lemma 2.1. *Assume $d \geq 3$ and conditions* (A0)–(A6)*. Then, we have:*
(i) *For any non-trivial function $u \in H^1(\mathbb{R}^d)$, there exists a unique $\lambda(u) > 0$ such that*

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{if} \quad 0 < \lambda < \lambda(u), \\ = 0 & \text{if} \quad \lambda = \lambda(u), \\ < 0 & \text{if} \quad \lambda > \lambda(u). \end{cases} \tag{2.7}$$

(ii)

$$\frac{d}{d\lambda}\mathcal{I}_\omega(T_\lambda u) > 0 \qquad \text{for any } \lambda > 0, \tag{2.8}$$

so that $\mathcal{I}_\omega(T_\lambda u)$ is monotone increasing with respect to $\lambda > 0$.
(iii)

$$\frac{d^2}{d^2\lambda}\mathcal{S}_\omega(T_\lambda u) \leq \frac{1}{\lambda^2}\mathcal{K}(T_\lambda u) \qquad \text{for any } \lambda > 0 \tag{2.9}$$

and $\mathcal{S}_\omega(T_\lambda u)$ is concave on $[\lambda(u), \infty)$.

Proof of Lemma 2.1. We see from (A3) and (A4) that $\mathcal{K}(T_\lambda u) > 0$ for any sufficiently small $\lambda > 0$. We also have

$$\mathcal{K}(T_\lambda u) \leq \lambda^2 \|\nabla u\|_{L^2}^2 - \lambda^{2^*} \|u\|_{L^{2^*}}^{2^*} \to -\infty \qquad \text{as } \lambda \to \infty. \tag{2.10}$$

Hence, there exists $\lambda_0 > 0$ such that $\mathcal{K}(T_{\lambda_0} u) = 0$. Since (2.6) shows $\frac{1}{\lambda^2}\mathcal{K}(T_\lambda u)$ is monotone decreasing with respect to λ, we find that (i) holds.

Using (2.4), we easily obtain (2.8).

A direct calculation together with (2.1), (2.2) and (A4) gives (2.9). Moreover, (2.9) together with (2.7) shows the concavity of $\mathcal{S}_\omega(T_\lambda u)$. □

3 Variational problems

In this section, we give the proofs of Proposition 1.1 and Theorem 1.1.

First, we prove Proposition 1.1:

Proof of Proposition 1.1. (i) We shall prove $m_\omega = \widetilde{m}_\omega$. Let $\{u_n\}$ be a minimizing sequence of the variational problem for $\widetilde{m}_\omega$, i.e., $\{u_n\}$ is a sequence in $H^1(\mathbb{R}^d) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \mathcal{I}_\omega(u_n) = \widetilde{m}_\omega, \tag{3.1}$$

$$\mathcal{K}(u_n) \leq 0 \quad \text{for any } n \in \mathbb{N}. \tag{3.2}$$

Then, it follows from (2.7) in Lemma 2.1 that for each $n \in \mathbb{N}$, there exists $\lambda_n \in (0, 1]$ such that $\mathcal{K}(T_{\lambda_n} u_n) = 0$. This together with (2.8) leads us to that

$$m_\omega \leq \mathcal{S}_\omega(T_{\lambda_n} u_n) = \mathcal{I}_\omega(T_{\lambda_n} u_n) \leq \mathcal{I}_\omega(u_n) = \widetilde{m}_\omega + o_n(1). \tag{3.3}$$

Hence, taking $n \to \infty$, we have $m_\omega \leq \widetilde{m}_\omega$. On the other hand, since

$$\widetilde{m}_\omega \leq \inf_{\substack{u\in H^1\setminus\{0\} \\ \mathcal{K}(u)=0}} \mathcal{I}_\omega(u) = \inf_{\substack{u\in H^1\setminus\{0\} \\ \mathcal{K}(u)=0}} \mathcal{S}_\omega(u) = m_\omega, \tag{3.4}$$

we have $\widetilde{m} \leq m$. Hence, it holds that $m_\omega = \widetilde{m}_\omega$.

(ii) We shall show that any minimizer of the variational problem for $\widetilde{m}_\omega$ is also a one for m_ω. Let Q_ω be a minimizer for $\widetilde{m}_\omega$, i.e., $Q_\omega \in H^1(\mathbb{R}^d)\setminus\{0\}$ with $\mathcal{K}(Q_\omega) \le 0$ and $\mathcal{I}_\omega(Q_\omega) = \widetilde{m}_\omega$. Since $\mathcal{I}_\omega = \mathcal{S}_\omega - \frac{1}{2}\mathcal{K}$ and $m_\omega = \widetilde{m}_\omega$, it is sufficient to show that $\mathcal{K}(Q_\omega) = 0$. Suppose the contrary that $\mathcal{K}(Q_\omega) < 0$. Then, it follows from (2.7) that there exists $0 < \lambda_0 < 1$ such that $\mathcal{K}(T_{\lambda_0} Q_\omega) = 0$. Moreover, we see from the definition of $\widetilde{m}_\omega$ and (2.8) that

$$\widetilde{m}_\omega \le \mathcal{I}(T_{\lambda_0} Q_\omega) < \mathcal{I}_\omega(Q_\omega) = \widetilde{m}_\omega, \tag{3.5}$$

which is a contradiction. Hence, $\mathcal{K}(Q_\omega) = 0$. □

Next, we give the prove of Theorem 1.1:

Proof of Theorem 1.1. In view of Proposition 1.1 (ii), it is sufficient to show the existence of a minimizer of the variational problem for $\widetilde{m}_\omega$. Let $\{u_n\}$ be a minimizing sequence for $\widetilde{m}_\omega$. We denote the Schwarz symmetrization of u_n by u_n^*. Then, we have

$$\mathcal{K}(u_n^*) \le 0 \quad \text{for any } n \in \mathbb{N}, \tag{3.6}$$

$$\lim_{n\to\infty} \mathcal{I}_\omega(u_n^*) = \widetilde{m}_\omega. \tag{3.7}$$

Besides, extracting some subsequence, we may assume that

$$\mathcal{I}_\omega(u_n^*) \le 1 + \widetilde{m}_\omega \qquad \text{for any } n \in \mathbb{N}, \tag{3.8}$$

which together with (1.8) gives us the boundedness of $\{u_n^*\}$ in $L^2(\mathbb{R}^d)$ and $L^{2^*}(\mathbb{R}^d)$:

$$\sup_{n\in\mathbb{N}} \|u_n^*\|_{L^2} \lesssim 1, \qquad \sup_{n\in\mathbb{N}} \|u_n^*\|_{L^{2^*}} \lesssim 1. \tag{3.9}$$

Moreover, using the Sobolev embedding, (3.6), the boundedness in L^2 and the growth conditions (A5) and (A6), we obtain that

$$\|u_n^*\|_{L^{2^*}}^2 \lesssim \|\nabla u_n^*\|_{L^2}^2 \lesssim \|u_n^*\|_{L^{p_1+1}}^{p_1+1} + \|u_n^*\|_{L^{2^*}}^{2^*} \lesssim \|u_n^*\|_{L^{2^*}}^{\frac{d(p_1-1)}{2}} + \|u_n^*\|_{L^{2^*}}^{2^*}. \tag{3.10}$$

This together with (3.9) shows that

$$\sup_{n\in\mathbb{N}} \|u_n^*\|_{H^1} \lesssim 1. \tag{3.11}$$

Now, since $\{u_n^*\}$ is radially symmetric and bounded in $H^1(\mathbb{R}^d)$, there exists a radially symmetric function $Q \in H^1(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} u_n^* = Q \quad \text{weakly in } H^1(\mathbb{R}^d), \tag{3.12}$$

$$\lim_{n\to\infty} u_n^* = Q \quad \text{strongly in } L^q(\mathbb{R}^d) \text{ for } 2 < q < 2^*, \tag{3.13}$$

$$\lim_{n\to\infty} u_n^*(x) = Q(x) \quad \text{for almost all } x \in \mathbb{R}^d, \tag{3.14}$$

$$\mathcal{I}_\omega(u_n^*) - \mathcal{I}_\omega(u_n^* - Q) - \mathcal{I}_\omega(Q) = o_n(1), \tag{3.15}$$

$$\mathcal{K}(u_n^*) - \mathcal{K}(u_n^* - Q) - \mathcal{K}(Q) = o_n(1). \tag{3.16}$$

This function Q is a candidate for the minimizer.

We shall first show that Q is non-trivial. Suppose the contrary that Q is trivial. Then, it follows from (3.6) and (3.13) that

$$0 \geq \limsup_{n\to\infty} \mathcal{K}(u_n^*) = 2\limsup_{n\to\infty} \left\{ \|\nabla u_n^*\|_{L^2}^2 - \|u^*\|_{L^{2^*}}^{2^*} \right\}, \tag{3.17}$$

so that

$$\limsup_{n\to\infty} \|\nabla u_n^*\|_{L^2}^2 \leq \liminf_{n\to\infty} \|u_n^*\|_{L^{2^*}}^{2^*} . \tag{3.18}$$

Moreover, this together with the definition of σ (see (1.14)) gives us that

$$\limsup_{n\to\infty} \|\nabla u_n^*\|_{L^2}^2 \geq \sigma \liminf_{n\to\infty} \|u_n^*\|_{L^{2^*}}^2 \geq \sigma \limsup_{n\to\infty} \|\nabla u_n^*\|_{L^2}^{\frac{2(d-2)}{d}} , \tag{3.19}$$

so that

$$\sigma^{\frac{d}{2}} \leq \limsup_{n\to\infty} \|\nabla u_n^*\|_{L^2}^2 \leq \liminf_{n\to\infty} \|u_n^*\|_{L^{2^*}}^{2^*} . \tag{3.20}$$

Hence, if Q is trivial, then we see from Proposition 1.1, (A3) and (3.20) that

$$m_\omega = \widetilde{m}_\omega = \lim_{n\to\infty} \mathcal{I}_\omega(u_n^*) \geq \liminf_{n\to\infty} \frac{1}{d} \|u_n^*\|_{L^{2^*}}^{2^*} \geq \frac{1}{d}\sigma^{\frac{d}{2}}. \tag{3.21}$$

However, Lemma 1.2 shows that this is a contradiction. Thus, Q is non-trivial.

Next, we shall show that $\mathcal{K}(Q) \leq 0$. Suppose the contrary that $\mathcal{K}(Q) > 0$. Then, it follows from (3.6) and (3.16) that $\mathcal{K}(u_n^* - Q) < 0$ for any sufficiently large $n \in \mathbb{N}$, so that (2.7) in Lemma 2.1 shows that there exists a unique $\lambda_n \in (0,1)$ such that $\mathcal{K}(T_{\lambda_n}(u_n^* - Q)) = 0$. Then, we see from (2.8) in Lemma 2.1 and (3.15) that

$$\begin{aligned} \widetilde{m}_\omega &\leq \mathcal{I}_\omega(T_{\lambda_n}(u_n^* - Q)) \\ &\leq \mathcal{I}_\omega(u_n^* - Q) = \mathcal{I}_\omega(u_n^*) - \mathcal{I}_\omega(Q) + o_n(1) \\ &= \widetilde{m}_\omega - \mathcal{I}_\omega(Q) + o_n(1). \end{aligned} \tag{3.22}$$

Hence, we conclude that $\mathcal{I}_\omega(Q) = 0$. However, this contradicts that Q is non-trivial. Thus, $\mathcal{K}(Q) \leq 0$.

Since Q is non-trivial and $\mathcal{K}(Q) \leq 0$, we see from the definition of $\widetilde{m}_\omega$ that

$$\widetilde{m}_\omega \leq \mathcal{I}_\omega(Q). \tag{3.23}$$

Moreover, it follows from (3.15) and (3.7) that

$$\mathcal{I}_\omega(Q) \leq \liminf_{n\to\infty} \mathcal{I}_\omega(u_n^*) \leq \widetilde{m}_\omega. \tag{3.24}$$

Combining (3.23) and (3.24), we obtain that $\mathcal{I}_\omega(Q) = m_\omega$. In particular, we have $m_\omega = \widetilde{m}_\omega > 0$. Thus, we have completed the proof. □

4 Well-posedness

In this section, we discuss the local well-posedness result in the energy critical case (see [8, 14, 20]).

Let us begin with the notion of admissible pairs: A pair of space-time indices $(q, r) \in [2,\infty] \times [2,\infty]$ is said to be L^2-admissible, if $\frac{1}{r} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$. In particular, $(2,\infty)$,

$(\frac{2(d+2)}{d}, \frac{2(d+2)}{d})$, $(\frac{2d(d+2)}{d^2+4}, \frac{2(d+2)}{d-2})$, $(\frac{2(d+2)(p-1)}{d(d+2)(p-1)-8}, \frac{(d+2)(p-1)}{2})$ and $(2^*, 2)$ are L^2-admissible pairs. For any L^2-admissible pair (q, r), we have

$$\left\|e^{it\Delta}u\right\|_{L^r(\mathbb{R},L^q)} \lesssim \|u\|_{L^2}\,, \tag{4.1}$$

and for any L^2-admissible pairs (q_1, r_1) and (q_2, r_2),

$$\left\|\int_{t_0}^{t} e^{i(t-t')\Delta}v(t')\,dt'\right\|_{L^{r_1}(\mathbb{R},L^{q_1})} \lesssim \|v\|_{L^{r_2'}(\mathbb{R},L^{q_2'})} \quad \text{for any } t_0 \in \mathbb{R}. \tag{4.2}$$

These estimates are called the Strichartz estimates.

Generally, a pair $(q, r) \in [2, \infty] \times [2, \infty]$ is called $\dot{H}^s$-admissible, if $\frac{1}{r} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{q} - \frac{s}{d}\right)$ (cf. [12, 17]). In particular, the pair $(\frac{(d+2)(p-1)}{2}, \frac{(d+2)(p-1)}{2})$ is the diagonal $\dot{H}^{s_p}$-admissible pair.

In order to mention the local well-posedness result, we need to introduce several space-time function spaces. Let I be an interval. Then, we introduce Strichartz-type function spaces:

$$S(I) := L^\infty(I, L^2) \cap L^2(I, L^{2^*}), \tag{4.3}$$

$$V_p(I) := L^{\frac{(d+2)(p-1)}{2}}(I, L^{\frac{2(d+2)(p-1)}{d(d+2)(p-1)-8}}), \tag{4.4}$$

$$W_p(I) := L^{\frac{(d+2)(p-1)}{2}}(I, L^{\frac{(d+2)(p-1)}{2}}), \tag{4.5}$$

$$V(I) := V_{1+\frac{4}{d-2}}(I) = L^{\frac{2(d+2)}{d-2}}(I, L^{\frac{2d(d+2)}{d^2+4}}), \tag{4.6}$$

$$W(I) := W_{1+\frac{4}{d-2}}(I) = L^{\frac{2(d+2)}{d-2}}(I, L^{\frac{2(d+2)}{d-2}}). \tag{4.7}$$

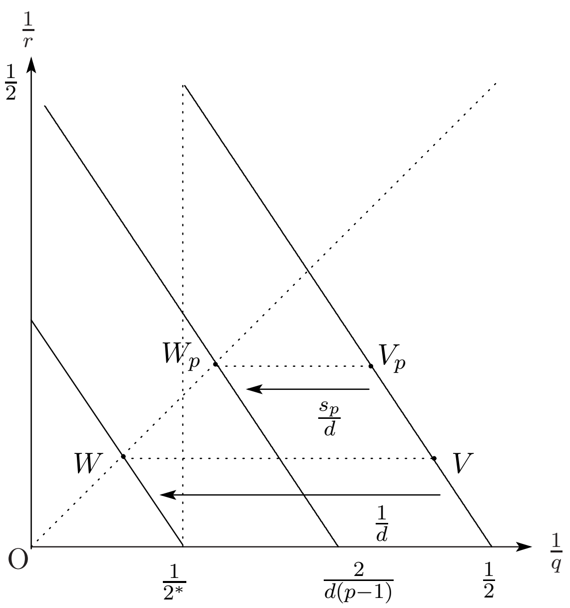

Figure 1: Strichartz type spaces

It is worthwhile noting that for any $1+\frac{4}{d}<p\le q\le 1+\frac{4}{d-2}$, we have

$$\left\||\nabla|^{s}\left(|u|^{q-1}u\right)\right\|_{L^{\frac{(d+2)(p-1)}{2p}}(I,L^{\frac{2d(d+2)(p-1)}{d(d+6)(p-1)-8}})}\lesssim \||\nabla|^{s}u\|_{V_p(I)}\|u\|_{W_q(I)}^{q-1}\quad \text{for } s=0,1 \tag{4.8}$$

and

$$\|u\|_{W_q(I)}\lesssim \||\nabla|^{s_q}u\|_{V_q(I)}\le \||\nabla|^{s_q}u\|_{L^\infty(I,L^2)}^{1-\frac{p-1}{q-1}}\||\nabla|^{s_q}u\|_{V_p(I)}^{\frac{p-1}{q-1}}. \tag{4.9}$$

Here, the Hölder conjugate of $\left(\frac{2d(d+2)(p-1)}{d(d+6)(p-1)-8},\frac{(d+2)(p-1)}{2p}\right)$ is L^2-admissible.

Theorem 4.1 (Well-posedness in H^1)**.** *Assume $d\ge 3$ and* (A0)–(A6)*. Then, we have:*

(i) *Let $\psi_0\in H^1(\mathbb{R}^d)$, I be an interval, $t_0\in I$ and $A>0$. Assume that*

$$\|\psi_0\|_{H^1}\le A. \tag{4.10}$$

Then, there exists $\delta>0$ depending on A with the following property; If

$$\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)}\le\delta, \tag{4.11}$$

then there exists a solution $\psi\in C(I,H^1(\mathbb{R}^d))$ to (NLS) such that

$$\psi(t_0)=\psi_0, \tag{4.12}$$

$$\|\langle\nabla\rangle\psi\|_{S(I)}\lesssim\|\psi_0\|_{H^1}, \tag{4.13}$$

$$\|\langle\nabla\rangle\psi\|_{V_{p_1}(I)}\le 2\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)}. \tag{4.14}$$

(ii) *Let I be an interval. Suppose that $\psi_1,\psi_2\in C(I,H^1(\mathbb{R}^d))$ are two solutions of (NLS) with $\psi_1(t_0)=\psi_2(t_0)$ for some $t_0\in I$. Then, $\psi_1=\psi_2$.*

Furthermore, let $\psi\in C(I_{\max},H^1(\mathbb{R}^d))$ be a solution to (NLS), where $I_{\max}$ is the maximal time interval on which the solution ψ exists (Note here that by (i)*, $I_{\max}$ must be open). Then, we have:*

(iii) *The following conservation laws hold; For any $t,t_0\in I_{\max}$,*

$$\mathcal{M}(\psi(t)):=\|\psi(t)\|_{L^2}^2=\mathcal{M}(\psi(t_0)), \tag{4.15}$$

$$\mathcal{H}(\psi(t))=\mathcal{H}(\psi(t_0)), \tag{4.16}$$

$$\mathcal{S}_\omega(\psi(t))=\mathcal{S}_\omega(\psi(t_0))\qquad \textit{for any } \omega\in\mathbb{R}, \tag{4.17}$$

$$\mathcal{P}(\psi(t)):=\Im\int_{\mathbb{R}^d}\nabla\psi(x,t)\overline{\psi(x,t)}\,dx=\mathcal{P}(\psi(t_0)). \tag{4.18}$$

(iv) *If $T_{\max}:=\sup I_{\max}<+\infty$, then*

$$\|\psi\|_{W_{p_1}([T,T_{\max}))\cap W([T,T_{\max}))}=\infty\qquad \textit{for any } T\in I_{\max}. \tag{4.19}$$

A similar result holds, when $T_{\min}:=\inf I_{\max}>-\infty$.

(v) *If*

$$\|\psi\|_{W_{p_1}(I_{\max})\cap W(I_{\max})} < \infty, \tag{4.20}$$

then $T_{\max} = +\infty$, $T_{\min} = -\infty$, *and there exist* $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ *such that*

$$\lim_{t\to+\infty} \left\|\psi(t) - e^{it\Delta}\phi_+\right\|_{H^1} = \lim_{t\to-\infty} \left\|\psi(t) - e^{it\Delta}\phi_-\right\|_{H^1} = 0. \tag{4.21}$$

Proof of Theorem 4.1. We first prove (i). Let $\delta > 0$ be a sufficiently small constant specified later; In particular, we take $\delta < 1$. Suppose that

$$\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)} \le \delta. \tag{4.22}$$

We see from the Strichartz estimate that there exists $C_{st} > 0$ depending only on d such that

$$\left\|e^{it\Delta}u\right\|_{S(\mathbb{R})} \le C_{st}\left\|u\right\|_{L^2}. \tag{4.23}$$

We define the space $Y(I)$ and the map $\mathcal{T}$ by

$$Y(I) := \left\{ u \in C(I, H^1(\mathbb{R}^d)) \,\middle|\, \begin{array}{l} \|\langle\nabla\rangle u\|_{V_{p_1}(I)} \le 2\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)}, \\ \|\langle\nabla\rangle u\|_{S(I)} \le 2C_{St}\left\|\psi_0\right\|_{H^1} \end{array} \right\}, \tag{4.24}$$

$$\mathcal{T}(u) := e^{i(t-t_0)\Delta}\psi_0 - i\int_{t_0}^{t} e^{i(t-t')\Delta}\left\{\, f(u) + |u|^{2^*-2}u \,\right\}(t')\,dt'. \tag{4.25}$$

We can verify that $Y(I)$ becomes a metric space with the metric $\rho(u,v) := \|u-v\|_{S(I)}$.

We shall show that $\mathcal{T}$ maps $Y(I)$ into itself for sufficiently small δ. The Strichartz estimate, together with (A5), (A6), (4.8), (4.9) and (4.22), gives us that

$$\begin{aligned} \|\langle\nabla\rangle\mathcal{T}(u)\|_{V_{p_1}(I)} &\le \left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)} \\ &\quad + C\left(1 + \|u\|_{L^\infty(I,H^1)}^{p_2-p_1} + \|u\|_{L^\infty(I,H^1)}^{2^*-(p_1+1)}\right)\|\langle\nabla\rangle u\|_{V_{p_1}(I)}^{p_1} \\ &\le \left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)} \\ &\quad + C(1 + A^{p_2-p_1} + A^{2^*-(p_1+1)})\delta^{p_1-1}\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)}, \end{aligned} \tag{4.26}$$

where C is some universal constant. Hence, taking δ sufficiently small depending on A, we obtain

$$\|\langle\nabla\rangle\mathcal{T}(u)\|_{V_{p_1}(I)} \le 2\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi_0\right\|_{V_{p_1}(I)}. \tag{4.27}$$

Similarly, taking δ sufficiently small depending on A, we have

$$\|\langle\nabla\rangle\mathcal{T}(u)\|_{S(I)} \le 2C_{St}\left\|\psi_0\right\|_{H^1}. \tag{4.28}$$

Next, we shall show that $\mathcal{T}$ is contraction in $(Y(I),\rho)$ for sufficiently small $\delta > 0$. Take any $u, v \in Y(I)$. Then, we have in a way similar to the estimate (4.26) that

$$\begin{aligned} \rho(\mathcal{T}(u),\mathcal{T}(v)) &= \|\mathcal{T}(u) - \mathcal{T}(v)\|_{S(I)} \\ &\lesssim (1 + A^{p_2-p_1} + A^{2^*-(p_1+1)})\delta^{p_1-1}\|u-v\|_{V_{p_1}(I)}, \end{aligned} \tag{4.29}$$

which shows that if δ is sufficiently small depending on A, then $\mathcal{T}$ is contraction.

Thus, the claim (i) follows from the contraction mapping principle in $(Y(I), \rho)$.

We omit the proof of (ii) and (iii).

We prove (iv) by contradiction. Assume $T_{\max} < +\infty$, and suppose the contrary that

$$\|\psi\|_{W_{p_1}([T_0, T_{\max}))\cap W([T_0, T_{\max}))} < \infty \quad \text{for some } T_0 \in I_{\max}. \tag{4.30}$$

In particular, we have

$$\lim_{T\to T_{\max}} \|\psi\|_{W_{p_1}([T, T_{\max}))\cap W_{p_2}([T, T_{\max}))\cap W([T, T_{\max}))} = 0. \tag{4.31}$$

In view of (i) in this theorem, it suffices to show that $\lim\limits_{t\to T_{\max}} \psi(t)$ exists strongly in $H^1(\mathbb{R}^d)$. We further reduce this to proving that for any sequence $\{t_n\}$ in $[T_0, T_{\max})$ with $\lim\limits_{n\to\infty} t_n = T_{\max}$, $\{e^{-it_n\Delta}\psi(t_n)\}$ is Cauchy sequence in $H^1(\mathbb{R}^d)$. Let us prove this. Put $p_3 := 1 + \frac{4}{d-2}$ as well as the above. The Strichartz estimate together with (4.8) yields that

$$\begin{aligned}
&\left\| e^{-it_n\Delta}\psi(t_n) - e^{-it_m\Delta}\psi(t_m) \right\|_{H^1} \\
&\le \sup_{t\in[t_m,t_n]} \left\| \int_{t_m}^{t} e^{-it'\Delta} \left\{ f(\psi) + |\psi|^{2^*-2}\psi \right\} (t')\, dt' \right\|_{H^1} \\
&\lesssim \|\langle\nabla\rangle\psi\|_{V_{p_1}([t_m, T_{\max}))} \sum_{j=1}^{3} \|\psi\|_{W_{p_j}([t_m, T_{\max}))}^{p_j-1}.
\end{aligned} \tag{4.32}$$

We see from (4.31) and (4.32) that $\{e^{-it_n\Delta}\psi(t_n)\}$ is Cauchy sequence in $H^1(\mathbb{R}^d)$, provided that

$$\|\langle\nabla\rangle\psi\|_{V_{p_1}([T, T_{\max}))} < \infty \quad \text{for some } T \in [T_0, T_{\max}). \tag{4.33}$$

We shall prove (4.33). By (4.31), we can take $T_1 \in [T_0, T_{\max})$ such that

$$\sum_{j=1}^{3} \|\psi\|_{W_{p_j}([T_1, T_{\max}))}^{p_j-1} \ll 1. \tag{4.34}$$

Then, an estimate similar to (4.32) gives us that

$$\|\langle\nabla\rangle\psi\|_{V_{p_1}([T_1, T_{\max}))} \le C\, \|\psi(T_1)\|_{H^1} + \frac{1}{2} \|\langle\nabla\rangle\psi\|_{V_{p_1}([T_1, T_{\max}))} \tag{4.35}$$

for some universal constant $C > 0$, from which we immediately obtain the desired result (4.33).

Next, we prove (v). We see from the contraposition of (iv) that $T_{\max} = +\infty$ and $T_{\min} = -\infty$. Moreover, an argument similar to the proof of (iv) shows that there exists $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that (4.21) holds. $\square$

Proposition 4.1 (see [8]). *Assume $d \ge 3$. Let $\phi \in \dot{H}^1(\mathbb{R}^d)$, let $t_0 \in \mathbb{R} \cup \{\pm\infty\}$ and let $A > 0$. Assume that*

$$\|\nabla\phi\|_{L^2} \le A. \tag{4.36}$$

Then, there exists $\delta > 0$ depending on A with the following property; If

$$\|\nabla\phi\|_{L^2(\mathbb{R})} \le \delta, \tag{4.37}$$

then there exists a global solution $\psi \in C(\mathbb{R}, \dot{H}^1(\mathbb{R}^d))$ *to* (NLS$_0$) *such that*

$$\begin{cases} \psi(t_0) = \phi & \text{if} \quad t_0 \in \mathbb{R}, \\ \lim_{t \to t_0} \left\| \psi(t) - e^{it\Delta}\phi \right\|_{H^1} = 0 & \text{if} \quad t_0 \in \{\pm\infty\}, \end{cases} \tag{4.38}$$

$$\|\nabla \psi\|_{S(\mathbb{R})} \lesssim \|\nabla \phi\|_{L^2}. \tag{4.39}$$

We can control the Strichartz spaces by a few spaces. Indeed, we have:

Lemma 4.2. *Assume* $d \ge 3$. *Let* I *be an interval,* $A, B > 0$, *and let* u *be a function such that*

$$\|u\|_{L^\infty(I,H^1)} \le A, \qquad \|u\|_{W_{p_1}(I) \cap W(I)} \le B. \tag{4.40}$$

Let $\varepsilon > 0$ *and suppose that*

$$\left\| \langle \nabla \rangle \left(i\frac{\partial}{\partial t}u + \Delta u + f(u) + |u|^{2^*-2}u \right) \right\|_{L^{\frac{2(d+2)}{d+4}}(I, L^{\frac{2(d+2)}{d+4}})} \le \varepsilon. \tag{4.41}$$

Then, we have

$$\|\langle \nabla \rangle u\|_{S(I)} \lesssim C(A,B) + \varepsilon, \tag{4.42}$$

where $C(A,B)$ *is some constant depending on* A *and* B.

Proof of Lemma 4.42. We rewrite the function u by

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^{t} e^{i(t-t')\Delta}\left(2i\frac{\partial u}{\partial t} + \Delta u\right)(t')\, dt \quad \text{for any } t_0 \in I, \tag{4.43}$$

where the equality is taken in the weak sense. Then, the Strichartz estimate together with (4.8) and (4.41) gives us that

$$\begin{aligned} \|\langle \nabla \rangle u\|_{S(I)} &\lesssim \|u(t_0)\|_{H^1} + \sum_{k=1}^{2} \|\langle \nabla \rangle u\|_{V_{p_k}(I)} \|u\|_{W_{p_k}(I)}^{p_k-1} + \|\langle \nabla \rangle u\|_{V(I)} \|u\|_{W(I)}^{\frac{4}{d-2}} + \varepsilon \\ &\le A + \sum_{j=1}^{2} \|\langle \nabla \rangle u\|_{V_{p_k}(I)} B^{p_k-1} + \|\langle \nabla \rangle u\|_{V(I)} B^{\frac{4}{d-2}} + \varepsilon. \end{aligned} \tag{4.44}$$

Here, we see from the Hölder inequality that

$$\begin{aligned} \|\langle \nabla \rangle u\|_{V_p(I)} &\le \|u\|_{L^\infty(I,H^1)}^{1-\frac{4}{(d+2)(p-1)}} \|\langle \nabla \rangle u\|_{L^2(I,L^{2^*})}^{\frac{4}{(d+2)(p-1)}} \\ &\le A^{1-\frac{4}{(d+2)(p-1)}} \|\langle \nabla \rangle u\|_{S(I)}^{\frac{4}{(d+2)(p-1)}} \qquad \text{for any } 1 + \tfrac{4}{d} < p \le 1 + \tfrac{4}{d-2}. \end{aligned} \tag{4.45}$$

Combining (4.44) with (4.45), we easily verify that the desired result (4.42) holds. □

5 Perturbation Theory

The derivative of our nonlinearity is no longer Lipschitz continuous when $d \geq 5$; It is merely Hölder continuous of order $p-1 > \frac{4}{d}$. Thus, to establish the long-time perturbation theory (see Propositions 5.6), we need some idea. We will employ the exotic Strichartz estimate (see [11] and [20]).

Assume that $d \geq 5$. We define α by $s_\alpha = 1 - \frac{4}{d}s_{p_1}$, i.e.,

$$\alpha := 1 + \frac{4d(p_1-1)}{d(d+2)(p_1-1)-16} \in \left(1 + \frac{4d}{d^2-2d+8},\ 1 + \frac{4}{d-2} \right). \tag{5.1}$$

Let (ρ, γ) and (ρ^*, γ^*) be the pairs such that

$$\rho := \frac{\alpha+1+2^*}{2}, \qquad \frac{1}{\gamma} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\rho} - \frac{s_\alpha}{d}\right), \tag{5.2}$$

$$\rho^* := \rho', \qquad \frac{1}{r^*} = 1 - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\rho} + \frac{s_\alpha}{d}\right). \tag{5.3}$$

Note here that α and ρ are monotone decreasing with respect to p_1.

Our exotic Strichartz norms are as follows[1]:

$$\|u\|_{ES(I)} := \left\| |\nabla|^{\frac{4}{d}s_{p_1}} u \right\|_{L^\gamma(I, L^\rho)}, \tag{5.4}$$

$$\|u\|_{ES^*(I)} := \left\| |\nabla|^{\frac{4}{d}s_{p_1}} u \right\|_{L^{\gamma^*}(I, L^{\rho^*})}. \tag{5.5}$$

Since (ρ, γ) are $\dot{H}^{1-\frac{4}{d}s_{p_1}}$-admissible ($\dot{H}^{s_\alpha}$-admissible) with $\frac{(d+2)(p_1-1)}{2} < \gamma < \infty$ $(d \geq 5)$, we see that

$$\|u\|_{ES(I)} \lesssim \|u\|_{L^\infty(I,H^1)}^{1-\theta_{ES}} \|\langle\nabla\rangle u\|_{V_{p_1}(I)}^{\theta_{ES}} \qquad \text{for some } 0 < \theta_{ES} < 1, \tag{5.6}$$

$$\|u\|_{L^\gamma(I, L^{\frac{d^2\rho}{d^2+d\rho-4\rho s_{p_1}}})} \lesssim \|u\|_{ES(I)}, \tag{5.7}$$

where $(\frac{d^2\rho}{d^2+d\rho-4\rho s_{p_1}}, \gamma)$ is L^2-admissible.

Lemma 5.1 (Exotic Strichartz estimate)**.** *Let I be an interval, $t_0 \in I$, and let u be a function on $\mathbb{R}^d \times I$. Then, we have*

$$\|u\|_{ES(I)} \leq \left\| e^{i(t-t_0)\Delta} u(t_0) \right\|_{ES(I)} + C \left\| i\frac{\partial u}{\partial t} + \Delta u \right\|_{ES^*(I)} \tag{5.8}$$

for some constant $C > 0$ depending only on d.

Proof of Lemma 5.1. We can write any function u by

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^{t} e^{i(t-t')\Delta}\left(i\frac{\partial u}{\partial t} + \Delta u\right)(t')\, dt, \tag{5.9}$$

where the equality is taken in the weak sense. Then, the claim follows from the inhomogeneous Strichartz estimate by Foschi [11]. $\square$

[1] When $d \leq 4$, we take $\|u\|_{ES(I)} = \|\nabla u\|_{V(I)}$ and $\|u\|_{ES^*(I)} = \|\nabla u\|_{L^{\frac{2(d+2)}{d+4}}(I, L^{\frac{2(d+2)}{d+4}})}$.

In order to treat the fractional differential operator $|\nabla|^{\frac{4}{d}s_{p_1}}$ in the exotic Strichartz norms, we employ the following estimates:

Lemma 5.2 (see [9])**.** *Let $s \in (0,1]$ and let $1 < q, q_1, q_2, q_3, q_4 < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}$. Then, we have*

$$\||\nabla|^s(uv)\|_{L^q} \lesssim \|u\|_{L^{q_1}} \||\nabla|^s v\|_{L^{q_2}} + \||\nabla|^s u\|_{L^{q_3}} \|v\|_{L^{q_4}}. \tag{5.10}$$

Lemma 5.3 (see [21])**.** *Let $h\colon \mathbb{C} \to \mathbb{C}$ be a Hölder continuous function of order $\alpha \in (0,1)$. Then, for any $s \in (0,\alpha)$, $q \in (1,\infty)$ and $\sigma \in (\frac{s}{\alpha}, 1)$, we have*

$$\||\nabla|^s h(u)\|_{L^q} \lesssim \|u\|^{\alpha - \frac{s}{\sigma}}_{L^{(\alpha-\frac{s}{\sigma})q_1}} \||\nabla|^\sigma u\|^{\frac{s}{\sigma}}_{L^{\frac{s}{\sigma}q_2}}, \tag{5.11}$$

provided that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $(\alpha - \frac{s}{\sigma})q_1 > \alpha$.

Lemma 5.4. *Assume that $d \geq 5$. Let h_1 and h_* be Hölder continuous functions of order $p_1 - 1$ and $\frac{4}{d-2}$, respectively. Let I be an interval. Then, we have*

$$\|h_1(v)w\|_{ES^*(I)} \lesssim \||\nabla|^{s_{p_1}} v\|^{p_1-1}_{L^{r_0}(I,L^{q_0})\cap L^{\widetilde{r}_0}(I,L^{\widetilde{q}_0})} \|w\|_{ES(I)}, \tag{5.12}$$

$$\|h_*(v)w\|_{ES^*(I)} \lesssim \|\nabla v\|^{\frac{4}{d-2}}_{L^{r_0}(I,L^{q_0})\cap L^{\widetilde{r}_0}(I,L^{\widetilde{q}_0})} \|w\|_{ES(I)}, \tag{5.13}$$

where (q_0, r_0) and $(\widetilde{q}_0, \widetilde{r}_0)$ are some L^2-admissible pairs with $\frac{(d+2)(p_1-1)}{2} < r_0, \widetilde{r}_0 < \infty$.

Proof of Lemma 5.4. Using Lemma 5.2, we obtain that

$$\begin{aligned}
&\|h_1(v)w\|_{ES^*(I)} \\
&\lesssim \|h_1(v)\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{\rho}{\rho-2}})} \|w\|_{ES(I)} \\
&\quad + \left\||\nabla|^{\frac{4}{d}s_{p_1}} h_1(v)\right\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{d^2(p_1-1)\rho}{d(d+2)(p_1-1)\rho-8\rho-2d^2(p_1-1)}})} \|w\|_{L^\gamma(I,L^{\frac{d^2\rho}{d^2-4s_{p_1}\rho}})}.
\end{aligned} \tag{5.14}$$

Since $\dot{W}^{\frac{4}{d}s_{p_1},\rho} \hookrightarrow L^{\frac{d^2\rho}{d^2-4s_{p_1}\rho}}$, we further obtain that

$$\begin{aligned}
\|h_1(v)w\|_{ES^*(I)} &\lesssim \|v\|^{p_1-1}_{L^{\frac{2(p_1-1)\rho}{2d-(d-2)\rho}}(I,L^{\frac{(p_1-1)\rho}{\rho-2}})} \|w\|_{ES(I)} \\
&\quad + \left\||\nabla|^{\frac{4}{d}s_{p_1}} h_1(v)\right\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{d^2(p_1-1)\rho}{d(d+2)(p_1-1)\rho-8\rho-2d^2(p_1-1)}})} \|w\|_{ES(I)}.
\end{aligned} \tag{5.15}$$

Here, $(\frac{(p_1-1)\rho}{\rho-2}, \frac{2(p_1-1)\rho}{2d-(d-2)\rho})$ is an $\dot{H}^{s_{p_1}}$-admissible pair in $(\frac{d(p_1-1)}{2}, \frac{(d+2)(p_1-1)}{2}) \times (\frac{(d+2)(p_1-1)}{2}, \infty)$. Moreover, it follows from Lemma 5.3 and the Hölder inequality in time that

$$\begin{aligned}
&\left\||\nabla|^{\frac{4}{d}s_{p_1}} h_1(v)\right\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{d^2(p_1-1)\rho}{d(d+2)(p_1-1)\rho-8\rho-2d^2(p_1-1)}})} \\
&\lesssim \|v\|^{p-1-\frac{4}{d}}_{L^{(p_1-1-\frac{4}{d})r_1}(I,L^{(p_1-1-\frac{4}{d})q_1})} \||\nabla|^{s_{p_1}} v\|^{\frac{4}{d}}_{L^{\frac{4}{d}r_2}(I,L^{\frac{4}{d}q_2})},
\end{aligned} \tag{5.16}$$

where

$$q_1 := \frac{d(d+2)(p_1-1)}{2d(p_1-1)-8}, \qquad q_2 := \frac{d^2(d+2)(p_1-1)\rho}{d(p_1-1)\{(d^2+2d+4)\rho-2(d^2+2d)\}-16\rho}, \tag{5.17}$$

$$r_1 := \frac{d(d+2)(p_1-1)}{2d(p_1-1)-8}, \qquad r_2 := \frac{2d(d+2)(p_1-1)\rho}{16\rho-d^2(p_1-1)\{d\rho-2(d+2)\}}. \tag{5.18}$$

Here, $((p_1-1-\frac{4}{d})q_1, (p_1-1-\frac{4}{d})r_1) = (\frac{(d+2)(p_1-1)}{2}, \frac{(d+2)(p_1-1)}{2})$ is a diagonal $\dot{H}^{s_{p_1}}$-admissible pair, and $(\frac{4}{d}q_2, \frac{4}{d}r_2)$ is an L^2-admissible pair in $(2, \frac{(d+2)(p_1-1)}{2}) \times (\frac{(d+2)(p_1-1)}{2}, \infty)$.

Combining the estimates (5.15) and (5.16), we obtain the desired estimate (5.12).

Next, we consider the estimate (5.13). Since $(\frac{4}{d-2}\frac{\rho}{\rho-2}, \frac{4}{d-2}\frac{\rho}{\rho-2})$ is $\dot{H}^1$-admissible, we see from the same estimate as the above that

$$\begin{aligned}\|h_*(v)w\|_{ES^*(I)} &\lesssim \|\nabla v\|_{L^{r_3}(I,L^{q_3})}^{\frac{4}{d-2}} \|w\|_{ES(I)} \\ &\quad + \left\||\nabla|^{\frac{4}{d}s_{p_1}} h_*(v)\right\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{d^2(p_1-1)\rho}{d(d+2)(p_1-1)\rho-8\rho-2d^2(p_1-1)}})} \|w\|_{ES(I)}\end{aligned} \tag{5.19}$$

for some L^2-admissible pair $(q_3, r_3) \neq (2,\infty), (2^*, 2)$. Moreover, it follows from Lemma 5.3 and the Hölder inequality in time that

$$\begin{aligned}&\left\||\nabla|^{\frac{4}{d}s_{p_1}} h_*(v)\right\|_{L^{\frac{2\rho}{2d-(d-2)\rho}}(I,L^{\frac{d^2(p_1-1)\rho}{d(d+2)(p_1-1)\rho-8\rho-2d^2(p_1-1)}})} \\ &\lesssim \|v\|_{L^{(\frac{4}{d-2}-\frac{4}{d})r_1^*}(I,L^{(\frac{4}{d-2}-\frac{4}{d})q_1^*})}^{\frac{4}{d-2}-\frac{4}{d}} \||\nabla|^{s_{p_1}} v\|_{L^{\frac{4}{d}r_2^*}(I,L^{\frac{4}{d}q_2^*})}^{\frac{4}{d}},\end{aligned} \tag{5.20}$$

where

$$q_1^* := \frac{d(d+2)}{4}, \qquad q_2^* := \frac{d^2(d+2)(p_1-1)\rho}{d^2(d+4)(p_1-1)\rho-8(d+2)\rho-2d^2(d+2)(p_1-1)}, \tag{5.21}$$

$$r_1^* := \frac{d(d+2)}{4}, \qquad r_2^* := \frac{2d(d+2)\rho}{2d^2(d+2)-d(d-2)(d+2)\rho-8\rho}. \tag{5.22}$$

Since $((\frac{4}{d-2}-\frac{4}{d})q_1^*, (\frac{4}{d-2}-\frac{4}{d})r_1^*) = (\frac{2(d+2)}{d-2}, \frac{2(d+2)}{d-2})$ is a diagonal $\dot{H}^1$-admissible pair and $(\frac{4}{d}q_2^*, \frac{4}{d}r_2^*)$ is an $\dot{H}^{1-s_{p_1}}$-admissible pair in $(2, 2^*) \times (2, \infty)$, the Sobolev embedding gives us the desired estimate (5.13). $\square$

Let (q_0, r_0) and $(\widetilde{r}_0, \widetilde{q}_0)$ be L^2-admissible pairs found in Lemma 5.4. In our proof of the perturbation theories below, we will need the auxiliary space

$$X(I) := L^{r_0}(I, L^{q_0}) \cap L^{\widetilde{r}_0}(I, L^{\widetilde{q}_0}). \tag{5.23}$$

Proposition 5.5 (Short-time perturbation theory, [20]). *Assume $d \geq 5$. Let I be an interval, $\psi \in C(I, H^1(\mathbb{R}^d))$ be a solution to (NLS), and let u be a function in $C(I, H^1(\mathbb{R}^d))$. Put*

$$e := i\frac{\partial u}{\partial t} + \Delta u + f(u) + |u|^{2^*-2}u. \tag{5.24}$$

Let $A_1, A_2 > 0$ and $t_0 \in I$, and assume that

$$\|\psi\|_{L^\infty(I,H^1)} \leq A_1, \tag{5.25}$$

$$\|\psi(t_0) - u(t_0)\|_{H^1} \leq A_2. \tag{5.26}$$

Then, there exists $\delta_0 > 0$ depending on A_1 and A_2 with the following property: If

$$\|\langle\nabla\rangle u\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \le \delta_0, \tag{5.27}$$

and

$$\|\langle\nabla\rangle e\|_{L^{\frac{2(d+2)}{d+4}}(I,L^{\frac{2(d+2)}{d+4}})} \le \delta, \tag{5.28}$$

$$\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\left(u(t_0)-\psi(t_0)\right)\right\|_{V_{p_1}(I)} \le \delta \tag{5.29}$$

for some $0<\delta\le\delta_0$, then we have

$$\|\langle\nabla\rangle\left(u-\psi\right)\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \lesssim \delta+\delta^{\frac{4}{d}\theta_{ES}}, \tag{5.30}$$

$$\left\|\langle\nabla\rangle\left(i\frac{\partial}{\partial t}+\Delta\right)(u-\psi)+\langle\nabla\rangle e\right\|_{L^2(I,L^{\frac{2d}{d+2}})} \lesssim \delta+\delta^{\frac{4}{d}\theta_{ES}}, \tag{5.31}$$

$$\|\langle\nabla\rangle\left(u-\psi\right)\|_{S(I)} \lesssim A_2+\delta+\delta^{\frac{4}{d}\theta_{ES}}, \tag{5.32}$$

$$\|\langle\nabla\rangle\psi\|_{S(I)} \lesssim A_1+A_2. \tag{5.33}$$

Proof of Proposition 5.5. Put

$$w := \psi - u. \tag{5.34}$$

Then, w satisfies that

$$\begin{aligned} i\frac{\partial}{\partial t}w+\Delta w = &-\left(\ f(w+u)-f(u)\ \right) \\ &-\left(\ |w+u|^{2^*-2}(w+u)-|u|^{2^*-2}u\ \right)-e. \end{aligned} \tag{5.35}$$

We can divide

$$\frac{\partial}{\partial z}\left(f(u)+|u|^{2^*-2}u\right)=g_1(u)+g_*(u), \tag{5.36}$$

$$\frac{\partial}{\partial \bar{z}}\left(f(u)+|u|^{2^*-2}u\right)=h_1(u)+h_*(u), \tag{5.37}$$

where g_1 and h_1 are Hölder continuous functions of order p_1-1, and g_* and h_* are ones of order $2^*-2=\frac{4}{d-2}$.

The exotic Strichartz estimate (Lemma 5.1) together with (5.6), (5.28), (5.29) and Lemma 5.4 shows

$$\begin{aligned} \|w\|_{ES(I)} &\lesssim A_2^{1-\theta_{ES}}\delta^{\theta_{ES}}+\left\|\int_0^1\left(g_1+h_1\right)(u+\theta w)\,d\theta\, w\right\|_{ES^*(I)} \\ &\quad+\left\|\int_0^1\left(g_*+h_*\right)(u+\theta w)\,d\theta\,\overline{w}\right\|_{ES^*(I)}+\|\langle\nabla\rangle e\|_{L^{\frac{2(d+2)}{d+4}}(I,L^{\frac{2(d+2)}{d+4}})} \\ &\lesssim A_2^{1-\theta_{ES}}\delta^{\theta_{ES}}+\left(\||\nabla|^{s_{p_1}}u\|_{X(I)}^{p_1-1}+\||\nabla|^{s_{p_1}}\psi\|_{X(I)}^{p_1-1}\right)\|w\|_{ES(I)} \\ &\quad+\left(\|\nabla u\|_{X(I)}^{2^*-2}+\|\nabla\psi\|_{X(I)}^{2^*-2}\right)\|w\|_{ES(I)}+\delta. \end{aligned} \tag{5.38}$$

We shall derive an estimate for $\|\langle\nabla\rangle\psi\|_{X(I)}$. Using the triangle inequality and (5.29), we have

$$\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi(t_0)\right\|_{V_{p_1}(I)} \le \left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}u(t_0)\right\|_{V_{p_1}(I)} + \delta. \tag{5.39}$$

Here, the Strichartz estimate together with (4.8), (5.27) and (5.28) gives us that

$$\begin{aligned}
&\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}u(t_0)\right\|_{V_{p_1}(I)} \\
&= \left\|\langle\nabla\rangle \left(u - i\int_{t_0}^{t} e^{i(t-t')\Delta}\left\{f(u)+|u|^{2^*-2}u+e\right\}(t')dt'\right)\right\|_{V_{p_1}(I)} \\
&\lesssim \delta_0 + \left(\delta_0^{p_1}+\delta_0^{p_2}+\delta_0^{2^*-1}\right)+\delta \lesssim \delta_0, \qquad \text{provided that } \delta_0 \le 1.
\end{aligned} \tag{5.40}$$

Combining (5.39) and (5.40), we have

$$\left\|\langle\nabla\rangle e^{i(t-t_0)\Delta}\psi(t_0)\right\|_{V_{p_1}(I)} \lesssim \delta_0. \tag{5.41}$$

Hence, taking δ_0 sufficiently small depending on A_1, we find from Theorem 4.1 that

$$\|\langle\nabla\rangle\psi\|_{V_{p_1}(I)} \lesssim \delta_0, \tag{5.42}$$

which together with the Sobolev embedding, the Hölder inequality and (5.25) also yields

$$\|\psi\|_{W_{p_1}(I)} \lesssim \delta_0, \tag{5.43}$$

$$\|\psi\|_{W(I)} \lesssim \|\langle\nabla\rangle\psi\|_{V(I)} \le A_1^{1-\theta_V}\delta_0^{\theta_V} \qquad \text{for some } 0<\theta_V<1, \tag{5.44}$$

$$\|\langle\nabla\rangle\psi\|_{X(I)} \lesssim A_1^{1-\theta_X}\delta_0^{\theta_X} \qquad \text{for some } 0<\theta_X<1. \tag{5.45}$$

Returning to (5.38), we see from (5.27) and (5.45) that

$$\begin{aligned}
\|w\|_{ES(I)} &\lesssim A_2^{1-\theta_{ES}}\delta^{\theta_{ES}} + \delta \\
&\quad + \left\{\ \delta_0^{p_1-1} + (A^{1-\theta_X}\delta_0)^{p_1-1} + \delta_0^{2^*-2} + (A^{1-\theta_X}\delta_0^{\theta_X})^{2^*-2}\ \right\}\|w\|_{ES(I)},
\end{aligned} \tag{5.46}$$

so that

$$\|w\|_{ES(I)} \lesssim A_2^{1-\theta_{ES}}\delta^{\theta_{ES}}, \tag{5.47}$$

provided that δ_0 is sufficiently small depending on A_1 and A_2.

Now, we shall show (5.30). Note that we have by (A5) and (A6) that

$$|f(w+u)-f(u)| \lesssim \left(|\psi|^{p_1-1}+|u|^{p_1-1}+|\psi|^{p_2-1}+|u|^{p_2-1}\right)|w| \tag{5.48}$$

and

$$\begin{aligned}
&|\nabla(f(w+u)-f(u))| \le |\nabla f(w+u)-\nabla f(u)| \\
&\lesssim \left(|\psi|^{p_1-1}+|\psi|^{p_2-1}\right)|\nabla w| + \left(|w|^{p_1-1}+|w|^{p_2-1}\right)|\nabla u|.
\end{aligned} \tag{5.49}$$

It follows from (5.29) and the inhomogeneous Strichartz estimate together with (5.48), (5.49), (5.7) and (5.28) that

$$
\begin{aligned}
&\|\langle\nabla\rangle w\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \\
&\lesssim \delta + \left(\|\psi\|_{W_{p_1}(I)}^{p_1-1} + \|\psi\|_{W_{p_2}(I)}^{p_2-1} + \|\psi\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle w\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \\
&\quad + \left(\|u\|_{W_{p_1}(I)}^{p_1-1} + \|u\|_{W_{p_2}(I)}^{p_2-1} + \|u\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle w\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \\
&\quad + \left(\|w\|_{ES(I)}^{p_1-1} + \|w\|_{ES(I)}^{p_2-1} + \|w\|_{ES(I)}^{2^*-2}\right) \|\nabla u\|_{S(I)} .
\end{aligned}
\tag{5.50}
$$

Here, the Strichartz estimate together with (5.25), (5.26), (4.8) and (5.28) yields that

$$
\|\langle\nabla\rangle u\|_{S(I)} \lesssim A_1 + A_2 + \left(\|u\|_{W_{p_1}(I)}^{p_1-1} + \|u\|_{W_{p_2}(I)}^{p_2-1} + \|u\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle u\|_{V_{p_1}(I)} + \delta. \tag{5.51}
$$

Hence, we see from (5.51) together with (5.27), and then taking δ_0 sufficiently small depending on A_1 and A_2, we have

$$
\|\langle\nabla\rangle u\|_{S(I)} \lesssim A_1 + A_2. \tag{5.52}
$$

Taking δ_0 sufficiently small depending on A_1 and A_2, we find from (5.50) together with (5.27), (5.43), (5.47) and (5.52) that

$$
\|\langle\nabla\rangle w\|_{V_{p_1}(I)\cap V(I)\cap X(I)} \lesssim \delta + \delta^{\frac{4}{d}\theta_{ES}}, \tag{5.53}
$$

which gives (5.30).

Next, we shall show (5.31). Using (5.35), (5.48), (5.49) and the Hölder inequality, we verify that

$$
\begin{aligned}
&\left\|\langle\nabla\rangle \left(i\frac{\partial}{\partial t} + \Delta\right)(u-\psi) + \langle\nabla\rangle e\right\|_{L^2(I, L^{\frac{2d}{d+2}})} \\
&\lesssim \left(\|\psi\|_{W_{p_1}(I)}^{p_1-1} + \|\psi\|_{W_{p_2}(I)}^{p_2-1} + \|\psi\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle w\|_{V(I)} \\
&\quad + \left(\|u\|_{W_{p_1}(I)}^{p_1-1} + \|u\|_{W_{p_2}(I)}^{p_2-1} + \|u\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle w\|_{V(I)} \\
&\quad + \left(\|w\|_{W_{p_1}(I)}^{p_1-1} + \|w\|_{W_{p_2}(I)}^{p_2-1} + \|w\|_{W(I)}^{2^*-2}\right) \|\langle\nabla\rangle u\|_{V(I)} .
\end{aligned}
\tag{5.54}
$$

Moreover, it follows from (5.54) together with (5.27), (5.43) and (5.53) that

$$
\left\|\langle\nabla\rangle \left(i\frac{\partial}{\partial t} + \Delta\right) w + \langle\nabla\rangle e\right\|_{L^2(I, L^{\frac{2d}{d+2}})} \lesssim \delta + \delta^{\frac{4}{d}\theta_{ES}}, \tag{5.55}
$$

provided that δ_0 is sufficiently small depending on A_1 and A_2. Hence, we have proved (5.31).

Finally. we prove (5.32) and (5.33). Combining the Strichartz estimate together with (5.26), (5.28) and (5.55), we obtain

$$
\|\langle\nabla\rangle w\|_{S(I)} \lesssim A_2 + \delta + \delta^{\frac{4}{d}\theta_{ES}}, \tag{5.56}
$$

so that (5.32) holds. Moreover, (5.33) follows from (5.52) and (5.56). ☐

Proposition 5.6 (Long-time perturbation theory)**.** *Assume $d \geq 3$ and $2_* - 1 < p < 2^* - 1$. Let I be an interval, $\psi \in C(I, H^1(\mathbb{R}^d))$ be a solution to (NLS), and let u be a function in $C(I, H^1(\mathbb{R}^d))$. Let $A_1, A_2, B > 0$ and $t_1 \in I$, and assume that*

$$\|\psi\|_{L^\infty(I,H^1)} \leq A_1, \tag{5.57}$$

$$\|\psi(t_1) - u(t_1)\|_{H^1} \leq A_2, \tag{5.58}$$

$$\|u\|_{W_{p_1}(I) \cap W(I)} \leq B. \tag{5.59}$$

Then, there exists $\delta > 0$ depending on A_1, A_2 and B such that if

$$\left\| \langle \nabla \rangle \left(i \frac{\partial}{\partial t} u + \Delta u + f(u) + |u|^{2^*-2} u \right) \right\|_{L^{\frac{2(d+2)}{d+4}}(I, L^{\frac{2(d+2)}{d+4}})} \leq \delta, \tag{5.60}$$

$$\left\| \langle \nabla \rangle e^{i(t-t_1)\Delta} \left(u(t_1) - \psi(t_1) \right) \right\|_{V_{p_1}(I)} \leq \delta \qquad \text{for some } t_1 \in I, \tag{5.61}$$

then we have

$$\|\langle \nabla \rangle \psi\|_{S(I)} < \infty. \tag{5.62}$$

Remark 5.1. *In the long-time perturbation theory above, we assume the uniform boundedness of ψ in $H^1(\mathbb{R}^d)$, instead of the one of u (cf. [13, 18]); For, it is easier to obtain the bound for ψ than the approximate solution u.*

Proof of Proposition 5.6. We consider the case $\inf I = t_1$ only. The other cases can be proven in the same way as this case.

Our first step is to derive a bound of $\|\langle \nabla \rangle u\|_{S(I)}$. Let $\eta > 0$ be a universal constant specified later, and assume $\delta \leq \eta$. We see from (5.59) that: there exist
(i) a number N' depending on B (and η), and
(ii) disjoint intervals $I_1', \dots, I_{N'}'$ of the form $I_j' = [t_j', t_{j+1}')$ ($t_1' = t_1$ and $t_{N+1}' := \max I$), such that

$$I = \bigcup_{j=1}^{N'} I_j', \tag{5.63}$$

$$\|u\|_{W_{p_1}(I_j') \cap W(I_j')} \leq \eta \qquad \text{for any } j = 1, \dots, N'. \tag{5.64}$$

Then, it follows from (5.60) and (5.64) that

$$\begin{aligned} \|\langle \nabla \rangle u\|_{S(I_j')} &\leq C_1 \left\| u(t_j') \right\|_{H^1} + C_2 \delta \\ &\quad + C_3 \left(\|u\|_{W_{p_1}(I_j')}^{p_1-1} + \|u\|_{W_{p_2}(I_j')}^{p_2-1} + \|u\|_{W(I_j')}^{2^*-2} \right) \|\langle \nabla \rangle u\|_{V(I_j')} \\ &\leq C_1 \left\| u(t_j') \right\|_{H^1} + C_2 \delta \\ &\quad + C_3' \left(\eta^{p_1-1} + \eta^{p_2-1} + \eta^{2^*-2} \right) \|\langle \nabla \rangle u\|_{S(I_j')} \qquad \text{for any } 1 \leq j \leq N', \end{aligned} \tag{5.65}$$

where C_1, C_2, C_3 and C_3' are some universal constants. We choose η so small that

$$C_3' \left(\eta^{p_1-1} + \eta^{p_2-1} + \eta^{2^*-2} \right) \leq \frac{1}{2}. \tag{5.66}$$

Here, (5.57) and (5.58) shows that

$$\left\|u(t_1')\right\|_{H^1} \le \|\psi\|_{L^\infty(I,H^1)} + \|u(t_1)-\psi(t_1)\|_{H^1} \le A_1 + A_2, \tag{5.67}$$

so that, taking δ so small that $C_2\delta \le A_1 + A_2$, we see from (5.65) together with (5.66) that

$$\|\langle\nabla\rangle u\|_{S(I_1')} \le 2(C_1+1)(A_1+A_2). \tag{5.68}$$

In particular, we have

$$\left\|u(t_2')\right\|_{H^1} \le 2(C_1+1)(A_1+A_2). \tag{5.69}$$

Hence, using (5.65) again, we obtain

$$\|\langle\nabla\rangle u\|_{S(I_2')} \le 2^2 C_1(C_1+1)(A_1+A_2) + 2(A_1+A_2) \lesssim A_1 + A_2. \tag{5.70}$$

Iterating this, we consequently have

$$\|\langle\nabla\rangle u\|_{S(I)} \le C(A_1, A_2, B) \tag{5.71}$$

for some constant $C(A_1,A_2,B)>0$ depending on A_1, A_2 and B. In particular, we have

$$\sup_{t\in I}\|\psi(t)-u(t)\|_{H^1} \le A_1 + C(A_1,A_2,B) =: A_2'. \tag{5.72}$$

Now, let δ_0 be the constant found in Theorem 5.5 which is determined by A_1 and A_2' given in (5.57) and (5.71), and suppose that $\delta < \min\{\delta_0, \eta\}$. Then, we see from (5.71) that there exist
(i) a number N depending A_1, A_2 and B, and
(ii) disjoint intervals $I_1, \dots, I_N$ of the form $I_j = [t_j, t_{j+1})$ ($t_{N+1} := \max I$),
such that

$$I = \bigcup_{j=1}^{N} I_j, \tag{5.73}$$

$$\|\langle\nabla\rangle u\|_{V_{p_1}(I_j)\cap V(I_j)\cap X(I_j)} \le \delta_0 \qquad \text{for any } j=1,\dots,N. \tag{5.74}$$

Proposition 5.5 together with (5.60) and (5.61) shows

$$\|\langle\nabla\rangle(u-\psi)\|_{V_{p_1}(I_1)\cap V(I_1)\cap X(I_1)} \le C_0\left\{\delta + \delta^{\frac{4}{d}\theta_{EX}}\right\}, \tag{5.75}$$

$$\left\|\langle\nabla\rangle\left(i\frac{\partial}{\partial t}+\Delta\right)(u-\psi) + \langle\nabla\rangle e\right\|_{L^2(I_1,L^{\frac{2d}{d+2}})} \le C_0\left\{\delta + \delta^{\frac{4}{d}\theta_{ES}}\right\}, \tag{5.76}$$

$$\|\langle\nabla\rangle(u-\psi)\|_{S(I_1)} \le C_0\left\{A_2' + \delta + \delta^{\frac{4}{d}\theta_{ES}}\right\}, \tag{5.77}$$

$$\|\langle\nabla\rangle\psi\|_{S(I_1)} \le C_0\left\{A_1 + A_2'\right\}, \tag{5.78}$$

where C_0 is some universal constant. Here, we have the formula

$$\begin{aligned} e^{i(t-t_{j+1})\Delta}\left(u(t_{j+1})-\psi(t_{j+1})\right) &= e^{i(t-t_j)\Delta}\left(u(t_j)-\psi(t_j)\right) \\ &\quad + i\int_{I_j} e^{i(t-t')\Delta}\left\{\left(i\frac{\partial}{\partial t}+\Delta\right)(u-\psi)+e\right\}(t')\,dt'. \end{aligned} \tag{5.79}$$

Using the Strichartz estimate, the formula (5.79), (5.61) and (5.76), we obtain that

$$\begin{aligned}
&\left\| \langle \nabla \rangle e^{i(t-t_2)\Delta} \left(u(t_2) - \psi(t_2)\right)\right\|_{V_{p_1}(I)} \\
&\le \left\| \langle \nabla \rangle e^{i(t-t_1)\Delta} \left(u(t_1) - \psi(t_1)\right)\right\|_{V_{p_1}(I)} + \left\| \langle \nabla \rangle \left(i\frac{\partial}{\partial t} + \Delta \right)(u-\psi) + \langle \nabla \rangle e \right\|_{L^2(I_1, L^{\frac{2d}{d+2}})} \\
&\le \delta + C_0 \left\{ \delta + \delta^{\frac{4}{d}\theta_{ES}} \right\}.
\end{aligned} \tag{5.80}$$

We choose δ so small that

$$\delta_1 := \delta + C_0 \left\{ \delta + \delta^{\frac{4}{d}\theta_{ES}} \right\} < \delta_0. \tag{5.81}$$

Then, it follows from Proposition 5.5 that

$$\|\langle \nabla \rangle (u-\psi)\|_{V_{p_1}(I_2)\cap V(I_2) \cap X(I_2)} \le C_0 \left\{ \delta_1 + \delta_1^{\frac{4}{d}\theta_{EX}} \right\}, \tag{5.82}$$

$$\left\| \langle \nabla \rangle \left(i\frac{\partial}{\partial t} + \Delta \right)(u-\psi) + \langle \nabla \rangle e \right\|_{L^2(I_2, L^{\frac{2d}{d+2}})} \le C_0 \left\{ \delta_1 + \delta_1^{\frac{4}{d}\theta_{ES}} \right\}, \tag{5.83}$$

$$\|\langle \nabla \rangle (u-\psi)\|_{S(I_2)} \le C_0 \left\{ A_2' + \delta_1 + \delta_1^{\frac{4}{d}\theta_{ES}} \right\}, \tag{5.84}$$

$$\|\langle \nabla \rangle \psi\|_{S(I_2)} \le C_0 \left\{ A_1 + A_2' \right\}. \tag{5.85}$$

Moreover, we see from the formula (5.79) together with (5.80) and (5.76) that

$$\begin{aligned}
&\left\| \langle \nabla \rangle e^{i(t-t_3)\Delta} \left(u(t_3) - \psi(t_3)\right)\right\|_{V_{p_1}(I)} \\
&\le \left\| \langle \nabla \rangle e^{i(t-t_2)\Delta} \left(u(t_2) - \psi(t_2)\right)\right\|_{V_{p_1}(I)} + \left\| \langle \nabla \rangle \left(i\frac{\partial}{\partial t} + \Delta \right)(u-\psi) + \langle \nabla \rangle e \right\|_{L^2(I_2, L^{\frac{2d}{d+2}})} \\
&\le \delta_1 + C_0 \left\{ \delta_1 + \delta_1^{\frac{4}{d}\theta_{ES}} \right\}.
\end{aligned} \tag{5.86}$$

Hence, taking further small δ such that

$$\delta_2 := \delta_1 + C_0 \left\{ \delta_1 + \delta_1^{\frac{4}{d}\theta_{ES}} \right\} < \delta_0, \tag{5.87}$$

we can employ Proposition 5.5 on the interval I_3. Repeating this procedure N times, we find that if δ is sufficiently small depending on A_1, A_2 and B, then we have

$$\|\langle \nabla \rangle \psi\|_{S(I_j)} \le C_0(A_1 + A_2') \qquad \text{for any } 1 \le j \le N. \tag{5.88}$$

Hence, we have

$$\|\psi\|_{W_{p_1}(I)}^{\frac{(d+2)(p_1-1)}{2}} = \sum_{j=1}^{N} \|\psi\|_{W_{p_1}(I_j)}^{\frac{(d+2)(p_1-1)}{2}} \le \sum_{j=1}^{N} C_0(A_1 + A_2') \le C_0(A_1 + A_2')N. \tag{5.89}$$

Similarly, we have

$$\|\psi\|_{W(I)}^{\frac{2(d+2)}{d-2}} \le C_0(A_1 + A_2')N. \tag{5.90}$$

Then, the desired result (5.62) follows from Lemma 4.2. □

6 Scattering result

In this section, we prove Theorem 1.2.

6.1 Analysis on $A_{\omega,+}$

We discuss basic properties of the set $A_{\omega,+}$.

We first observe a relation between $\mathcal{K}$ and $\mathcal{H}$:

Lemma 6.1. *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ with*

$$\mathcal{K}(u_n) \ge 0 \qquad \textit{for any } n \in \mathbb{N}. \tag{6.1}$$

Then, we have

$$\mathcal{H}(u_n) > 0 \qquad \textit{for any } n \in \mathbb{N}. \tag{6.2}$$

Furthermore, if the sequence $\{u_n\}$ satisfies that

$$\liminf_{n\to\infty} \|\nabla u_n\|_{L^2} > 0, \tag{6.3}$$

then

$$\liminf_{n\to\infty} \mathcal{H}(u_n) > 0. \tag{6.4}$$

Proof of Lemma 6.1. Lemma 2.1 together with (6.1) shows that

$$\mathcal{K}(T_\lambda u_n) > 0 \quad \text{for any } \lambda \in (0,1) \text{ and } n \in \mathbb{N}. \tag{6.5}$$

Hence, we see from the relation (2.1) that

$$\mathcal{H}(u_n) > \mathcal{H}(T_\lambda u_n) \qquad \text{for any } \lambda \in (0,1) \text{ and } n \in \mathbb{N}. \tag{6.6}$$

Here, it follows from (A5), (A6) and the Sobolev embedding that there exists a constant $C > 0$ independent of n and λ such that

$$\begin{aligned}\mathcal{H}(T_\lambda u_n) &= \lambda^2 \|\nabla u_n\|_{L^2}^2 - F(T_\lambda u_n(x)) - \frac{1}{2^*}\|u_n\|_{L^{2^*}}^{2^*} \\ &\ge \lambda^2 \|\nabla u_n\|_{L^2}^2 - C\left(\sum_{j=1}^{2} \lambda^{\frac{d}{2}(p_j-1)} \|u_n\|_{H^1}^{p_j+1} + \lambda^{2^*}\|u_n\|_{H^1}^{2^*}\right).\end{aligned} \tag{6.7}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$ and $2 < \frac{d}{2}(p_1-1) < \frac{d}{2}(p_2-1) < 2^*$, we obtain the conclusions from the estimates (6.6) and (6.7). □

Next, we show that for each $\omega > 0$, $A_{\omega,+}$ is bounded in $H^1(\mathbb{R}^d)$:

Lemma 6.2. *Let $\omega > 0$, $m > 0$ and let u be a function in $H^1(\mathbb{R}^d)$. Assume that*

$$\mathcal{K}(u) \geq 0, \qquad \mathcal{S}_\omega(u) \leq m. \tag{6.8}$$

Then, we have

$$\|u\|_{L^2}^2 \leq \frac{2m}{\omega}, \qquad \|\nabla u\|_{L^2}^2 \lesssim m + \frac{m}{\omega}. \tag{6.9}$$

In particular, we have

$$\sup_{u \in A_{\omega,+}} \|u\|_{H^1}^2 \lesssim m_\omega + \frac{m_\omega}{\omega}. \tag{6.10}$$

Proof of Lemma 6.2. It follows from $\mathcal{K}(u) \geq 0$ that

$$\|u\|_{L^{2^*}}^{2^*} \leq \|\nabla u\|_{L^2}^2 , \tag{6.11}$$

$$\mathcal{I}_\omega(u) \leq \mathcal{S}_\omega(u) \leq m. \tag{6.12}$$

We see from (6.12) that

$$\|u\|_{L^2}^2 \leq \frac{2m}{\omega}, \tag{6.13}$$

which gives the first claim in (6.9).

Using (A5), (A6) and (6.11), we obtain that

$$m \geq \mathcal{S}_\omega(u) \geq \mathcal{H}(u) \geq \frac{1}{d} \|\nabla u\|_{L^2}^2 - C_1 \|u\|_{L^{p_1+1}}^{p_1+1} - C_2 \|u\|_{L^{p_2+1}}^{p_2+1} , \tag{6.14}$$

where C_1 and C_2 are universal positive constants. Moreover, using the Hölder inequality, (6.11) and (6.13), we have

$$\begin{aligned} m &\geq \frac{1}{d} \|\nabla u\|_{L^2}^2 - C_1 \|u\|_{L^2}^{\frac{(d+2)-(d-2)p_1}{2}} \|u\|_{L^{2^*}}^{\frac{d(p_1-1)}{2}} \\ &\qquad\qquad - C_2 \|u\|_{L^2}^{\frac{(d+2)-(d-2)p_2}{2}} \|u\|_{L^{2^*}}^{\frac{d(p_2-1)}{2}} \\ &\geq \frac{1}{d} \|\nabla \psi(t)\|_{L^2}^2 - C_1 \left(\frac{2m_\omega}{\omega}\right)^{\frac{(d+2)-(d-2)p_1}{4}} \left(\|\nabla \psi(t)\|_{L^2}^2\right)^{\frac{(d-2)(p_1-1)}{4}} \\ &\qquad\qquad - C_2 \left(\frac{2m_\omega}{\omega}\right)^{\frac{(d+2)-(d-2)p_2}{2}} \left(\|\nabla \psi(t)\|_{L^2}^2\right)^{\frac{(d-2)(p_2-1)}{4}} . \end{aligned} \tag{6.15}$$

Since $\frac{(d-2)(p_1-1)}{4} \leq \frac{(d-2)(p_2-1)}{4} < 1$, (6.15) together with the Young inequality yields the second claim in (6.9). $\square$

For a function $u \in H^1(\mathbb{R}^d)$ with $\mathcal{K}(u) \geq 0$, we can compare $\mathcal{H}(u)$ and $\|\nabla u\|_{L^2}^2$:

Lemma 6.3. *Assume $d \geq 3$ and conditions* (A0)–(A6)*. Let $u \in H^1(\mathbb{R}^d)$ be a function with $\mathcal{K}(u) \geq 0$. Then, we have*

$$\|\nabla u\|_{L^2}^2 \lesssim \mathcal{H}(u). \tag{6.16}$$

Proof of Lemma 6.3. Using the assumption (A3), we have

$$\int_{\mathbb{R}^d} (DF - 2F)(u) \geq \left(\frac{4}{d} + \varepsilon_0\right) \int_{\mathbb{R}^d} F(u) \geq 0. \tag{6.17}$$

Put $C_0' := \max\{\frac{4}{4+d\varepsilon_0}, \frac{d-2}{d}\}$. Clearly, $C_0' < 1$. Then, it follows from (6.17) and $\mathcal{K}(u) \geq 0$ that

$$\begin{aligned}
\mathcal{H}(u) &= \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}^d} F(u) - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*} \\
&\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}\left(\frac{d}{4+d\varepsilon_0}\right)\int_{\mathbb{R}^d} (DF - 2F)(\psi(t)) - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*} \qquad (6.18) \\
&\geq \frac{1}{2}\left(1 - C_0'\right)\|\nabla u\|_{L^2}^2 ,
\end{aligned}$$

which gives the desired result. ∎

The following lemma tells us that $A_{\omega,+}$ is invariant under the flow defined by (NLS). Strongly, $\mathcal{K}$ of a solution in $A_{\omega,+}$ is positive uniformly in time:

Lemma 6.4. *Let ψ be a solution to* (NLS) *starting from $A_{\omega,+}$, and let $I_{\max}$ be the maximal interval where ψ exists. Then, we have*

$$\psi(t) \in A_{\omega,+} \qquad \textit{for any } t \in I_{\max}, \tag{6.19}$$

$$\inf_{t\in I_{\max}} \mathcal{K}(\psi(t)) > 0. \tag{6.20}$$

Proof of Lemma 6.4. The claim (6.19) easily follows from the action conservation law and the definition of m_ω.

We shall prove (6.20). Put

$$\widetilde{F}(u) := F(u) + \frac{d-2}{d}|u|^{2^*}. \tag{6.21}$$

Then, we have

$$(D - 2_* - \widetilde{\varepsilon}_0)\widetilde{F} \geq 0, \tag{6.22}$$

$$(D-2)(D - 2_* - \widetilde{\varepsilon}_0)\widetilde{F} \geq 0, \tag{6.23}$$

where $\widetilde{\varepsilon}_0 := \min\{\varepsilon_0, \frac{2}{d-2}\}$. Let ψ be a solution to (NLS) starting from $A_{\omega,+}$. It is easy to verify that

$$\begin{aligned}
&\frac{d^2}{d\lambda^2}\mathcal{S}_\omega(T_\lambda\psi(t)) \\
&= -\frac{1}{\lambda^2}\mathcal{K}(T_\lambda\psi(t)) + \frac{2}{\lambda^2}\|\nabla T_\lambda\psi(t)\|_{L^2}^2 - \frac{d}{2\lambda^2}\int_{\mathbb{R}^d}\frac{d}{4}(D^2\widetilde{F} - 4D\widetilde{F} + 4\widetilde{F})(T_\lambda u).
\end{aligned} \tag{6.24}$$

Combining (6.24) with (6.23), we obtain

$$
\begin{aligned}
&\frac{d^2}{d\lambda^2}\mathcal{S}_\omega(T_\lambda\psi(t)) \\
&\le -\frac{1}{\lambda^2}\mathcal{K}(T_\lambda\psi(t)) + \frac{2}{\lambda^2}\|\nabla T_\lambda\psi(t)\|_{L^2}^2 - \frac{d}{2\lambda^2}\int_{\mathbb{R}^d}\left(1+\frac{d\widetilde{\varepsilon}_0}{4}\right)(D\widetilde{F}-2\widetilde{F})(T_\lambda\psi(t)) \\
&= -\frac{1}{\lambda^2}\mathcal{K}(T_\lambda\psi(t)) + \frac{2}{\lambda^2}\left\{\mathcal{K}(T_\lambda\psi(t)) - \frac{d}{4}\int_{\mathbb{R}^d}\frac{d\widetilde{\varepsilon}_0}{4}(D\widetilde{F}-2\widetilde{F})(T_\lambda\psi(t))\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } \lambda>0 \text{ and } t\in I_{\max}.
\end{aligned}
\tag{6.25}
$$

Suppose here that

$$
\mathcal{K}(\psi(t)) - \frac{d}{4}\int_{\mathbb{R}^d}\frac{d\widetilde{\varepsilon}_0}{4}(D\widetilde{F}-2\widetilde{F})(\psi(t)) \ge 0 \qquad \text{for any } t\in I_{\max}. \tag{6.26}
$$

Then, we have

$$
\left(1+\frac{d\widetilde{\varepsilon}_0}{4}\right)\mathcal{K}(\psi(t)) \ge \frac{d\widetilde{\varepsilon}_0}{4}\|\nabla\psi(t)\|_{L^2}^2 \gtrsim \mathcal{H}(\psi(t)) \gtrsim 1 \qquad \text{for any } t\in I_{\max}, \tag{6.27}
$$

where we have used the Hamiltonian conservation law and Lemma 6.1 to derive the final inequality. Thus, (6.20) holds in this case.

On the other hand, suppose that there exists $t_0\in I_{\max}$ such that

$$
\mathcal{K}(\psi(t_0)) - \frac{d}{4}\int_{\mathbb{R}^d}\frac{d\widetilde{\varepsilon}_0}{4}(D\widetilde{F}-2\widetilde{F})(\psi(t_0)) < 0. \tag{6.28}
$$

We see from the Sobolev embedding, (6.28), (A5), (A6) and the Hölder inequality that

$$
\|\psi(t_0)\|_{L^{2^*}}^2 \lesssim \|\nabla\psi(t_0)\|_{L^2}^2 \lesssim \sum_{j=1}^{2}\|\psi(t_0)\|_{L^2}^{p_j+1-\frac{d(p_j-1)}{2}}\|\psi(t_0)\|_{L^{2^*}}^{\frac{d(p_j-1)}{2}} + \|\psi(t_0)\|_{L^{2^*}}^{2^*}, \tag{6.29}
$$

which together with the mass conservation law gives us that

$$
\|\psi(t_0)\|_{L^{2^*}} \gtrsim 1. \tag{6.30}
$$

Let $\lambda(t_0)$ be a number such that

$$
\mathcal{K}(T_{\lambda(t_0)}\psi(t_0)) = 0. \tag{6.31}
$$

Then, Lemma 6.2, (6.31) and (6.30) show that

$$
\lambda(t_0)^2 \gtrsim \lambda(t_0)^2\|\nabla\psi(t_0)\|_{L^2}^2 \ge \lambda(t_0)^{2^*}\|\psi(t_0)\|_{L^{2^*}}^{2^*} \gtrsim \lambda(t_0)^{2^*}. \tag{6.32}
$$

Hence, we have

$$
\lambda(t_0) \lesssim 1. \tag{6.33}
$$

Now, we see from (6.28) and (2.6) that

$$
\begin{aligned}
&\frac{1}{\lambda^2}\left\{\mathcal{K}(T_\lambda\psi(t_0)) - \frac{d}{4}\int_{\mathbb{R}^d}\frac{d\widetilde{\varepsilon}_0}{4}(D\widetilde{F}-2\widetilde{F})(T_\lambda\psi(t_0))\right\} \\
&= \|\nabla\psi(t_0))\|_{L^2}^2 - \frac{d}{4}\left(1+\frac{d\widetilde{\varepsilon}_0}{4}\right)\frac{1}{\lambda^2}\int_{\mathbb{R}^d}(D\widetilde{F}-2\widetilde{F})(T_\lambda\psi(t_0)) < 0 \quad \text{for any } \lambda\ge 1.
\end{aligned}
\tag{6.34}
$$

Hence, (6.25) together with (2.7) in Lemma 2.1 and (6.34) shows

$$\frac{d^2}{d\lambda^2}\mathcal{S}_\omega(T_\lambda\psi(t_0)) < -\frac{1}{\lambda^2}\mathcal{K}(T_\lambda\psi(t_0)) < 0 \quad \text{for any } 1 \le \lambda \le \lambda(t_0). \tag{6.35}$$

Combining (6.35) with (6.33), we obtain that

$$\begin{aligned}\mathcal{K}(\psi(t_0)) &\gtrsim (\lambda(\psi(t_0)) - 1)\mathcal{K}(\psi(t_0)) = (\lambda(\psi(t_0)) - 1)\frac{d}{d\lambda}\mathcal{S}_\omega(T_\lambda\psi(t_0))\Big|_{\lambda=1} \\ &\ge \mathcal{S}_\omega(T_{\lambda(t_0)}\psi(t_0)) - \mathcal{S}_\omega(\psi(t_0)) \ge m_\omega - \mathcal{S}_\omega(\psi(t_0)) \gtrsim 1.\end{aligned} \tag{6.36}$$

This completes the proof. $\square$

6.2 Extraction of critical element

In view of Theorem 4.1 (v), it suffices for Theorem 1.2 to show that any solution ψ to (NLS) starting from $A_{\omega,+}$ satisfies $\|\psi\|_{W_{p_1}(I_{\max})\cap W(I_{\max})} < \infty$, where $I_{\max}$ denotes the maximal interval where ψ exists. To this end, for $m > 0$, we put

$$\tau_\omega(m) := \sup\left\{ \|\psi\|_{W_{p_1}(I_{\max})\cap W(I_{\max})} : \begin{array}{l} \psi \text{ is a solution to (NLS) such that} \\ \psi \in A_{\omega,+} \text{ and } \mathcal{S}_\omega(\psi) \le m \end{array} \right\} \tag{6.37}$$

and define

$$m_\omega^* := \sup\{m > 0 \colon \tau_\omega(m) < \infty\}. \tag{6.38}$$

These quantities were used in [13, 18]. It follows from the existence of a ground state for (1.4) that $m_\omega^* \le m_\omega$.

Our aim is to show that $m_\omega^* = m_\omega$. Here, let ψ be a solution to (NLS) such that $\psi \in A_{\omega,+}$ and $\mathcal{S}_\omega(\psi) \le m$. If m is sufficiently small, then Lemma 6.2 shows $\|\psi(t)\|_{H^1} \ll 1$. Hence, we see from Theorem 4.1 (i) that $m_\omega^* > 0$.

Now, we suppose the contrary that $m_\omega^* < m_\omega$. Then, we shall show the existence of the so-called critical element. To this end, we employ the following result for the equation (NLS$_0$) (see Corollary 1.9 in [18]. See also [16]), which causes the restriction $d \ge 5$ in Theorem 1.2:

Theorem 6.1. *Assume $d \ge 5$. Put*

$$A_0 := \left\{ u \in \dot{H}^1(\mathbb{R}^d) \,\middle|\, \mathcal{H}_0(u) < \frac{1}{d}\sigma^{\frac{d}{2}},\ \|\nabla u\|_{L^2}^2 < \sigma^{\frac{d}{2}} \right\}. \tag{6.39}$$

Then, any solution ψ to (NLS$_0$) *starting from A_0 exists globally in time and satisfies*

$$\psi(t) \in A_0 \quad \textit{for any } t \in \mathbb{R}, \qquad \sup_{t\in\mathbb{R}} \|\nabla\psi(t)\|_{L^2}^2 \le \sigma^{\frac{d}{2}} \tag{6.40}$$

and

$$\|\psi\|_{W(\mathbb{R})} < \infty. \tag{6.41}$$

Under the hypothesis $m_\omega^* < m_\omega$, we can take a sequence $\{\psi_n\}$ of solutions to (NLS) such that

$$\psi_n(t) \in A_{\omega,+} \qquad \text{for any } t \in I_n, \tag{6.42}$$

$$\lim_{n\to\infty} \mathcal{S}_\omega(\psi_n) = m_\omega^*, \tag{6.43}$$

$$\|\psi_n\|_{W_{p_1}(I_n)\cap W(I_n)} = \infty, \tag{6.44}$$

where I_n denotes the maximal interval where ψ_n exists (by time-translation, we may assume that each I_n contains 0). We also see from Lemma 6.2 that

$$\sup_{n\in\mathbb{N}} \|\psi_n(t)\|^2_{L^\infty(I_n,H^1)} \lesssim m_\omega + \frac{m_\omega}{\omega}. \tag{6.45}$$

We apply the profile decomposition (see Theorem 1.6 in [17]) to the sequence $\{\ |\nabla|^{-1}\langle\nabla\rangle\psi_n(0)\ \}$ and obtain some subsequence of $\{\psi_n(0)\}$ (still denoted by the same symbol) with the following property: there exists
(i) a family $\{\widetilde{u}^1, \widetilde{u}^2, \ldots\}$ of functions in $H^1(\mathbb{R}^d)$ (each $\widetilde{u}^j$ is called the *linear profile*),
(ii) a family $\{\ \{(x_n^1, t_n^1, \lambda_n^1)\}, \{(x_n^2, t_n^2, \lambda_n^2)\}, \ldots\ \}$ of sequences in $\mathbb{R}^d \times \mathbb{R} \times (0,1]$ with

$$\lim_{n\to\infty} t_n^j = t_\infty^j \in \mathbb{R} \cup \{\pm\infty\}, \tag{6.46}$$

$$\lim_{n\to\infty} \lambda_n^j = \lambda_\infty^j \in \{0,1\}, \qquad \lambda_n^j \equiv 1 \quad \text{if } \lambda_\infty^j = 1, \tag{6.47}$$

$$\lim_{n\to\infty} \left\{ \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\left|x_n^j - x_n^{j'}\right|}{\lambda_n^j} + \frac{\left|t_n^j - t_n^{j'}\right|}{\left(\lambda_n^j\right)^2} \right\} = +\infty \quad \text{for any } j' \neq j, \tag{6.48}$$

(iii) a family $\{w_n^1, w_n^2, \ldots\}$ of functions in $H^1(\mathbb{R}^d)$ with

$$\lim_{j\to\infty}\lim_{n\to\infty} \left\||\nabla|^{-1}\langle\nabla\rangle e^{it\Delta} w_n^j\right\|_{L^r(\mathbb{R},L^q)} = 0 \qquad \text{for any } \dot{H}^1\text{-admissible pair } (q,r), \tag{6.49}$$

such that, defining the transformations g_n^j and G_n^j by

$$(g_n^j u)(x) := \frac{1}{(\lambda_n^j)^{\frac{d-2}{2}}} u\left(\frac{x - x_n^j}{\lambda_n^j}\right), \tag{6.50}$$

$$(G_n^j v)(x,t) := \frac{1}{(\lambda_n^j)^{\frac{d-2}{2}}} v\left(\frac{x - x_n^j}{\lambda_n^j}, \frac{t - t_n^j}{(\lambda_n^j)^2}\right), \tag{6.51}$$

we have

$$\begin{aligned} e^{it\Delta}\psi_n(0) &= \sum_{j=1}^k \langle\nabla\rangle^{-1}|\nabla| G_n^j \left(\ e^{it\Delta}|\nabla|^{-1}\langle\nabla\rangle\widetilde{u}^j\ \right) + e^{it\Delta}w_n^k \\ &= \sum_{j=1}^k G_n^j \left(\frac{\langle(\lambda_n^j)^{-1}\nabla\rangle^{-1}\langle\nabla\rangle}{\lambda_n^j} e^{it\Delta}\widetilde{u}^j \right) + e^{it\Delta}w_n^k \\ &= \sum_{j=1}^k \frac{\langle\nabla\rangle^{-1}\langle\lambda_n^j\nabla\rangle}{\lambda_n^j} e^{i(t-t_n^j)\Delta} g_n^j \widetilde{u}^j + e^{it\Delta}w_n^k \qquad \text{for any } k \in \mathbb{N}. \end{aligned} \tag{6.52}$$

Note here that for any Fourier multiplier $\mu(\nabla)$ and the L^2-scaling operator T_λ, we have

$$\mu(\nabla)T_\lambda = T_\lambda \mu(\lambda\nabla). \tag{6.53}$$

Besides, putting

$$\sigma_n^j := \frac{\langle(\lambda_n^j)^{-1}\nabla\rangle^{-1}\langle\nabla\rangle}{\lambda_n^j}, \tag{6.54}$$

for any $k \in \mathbb{N}$ and $s = 0, 1$, we have the expansions:

$$\lim_{n\to\infty}\left\{\||\nabla|^s\psi_n(0)\|_{L^2}^2 - \sum_{j=1}^{k}\left\||\nabla|^s g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right\|_{L^2}^2 - \left\||\nabla|^s w_n^k\right\|_{L^2}^2\right\} = 0, \quad (6.55)$$

$$\lim_{n\to\infty}\left\{\mathcal{H}(\psi_n(0)) - \sum_{j=1}^{k}\mathcal{H}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) - \mathcal{H}(w_n^k)\right\} = 0, \quad (6.56)$$

$$\lim_{n\to\infty}\left\{\mathcal{S}_\omega(\psi_n(0)) - \sum_{j=1}^{k}\mathcal{S}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) - \mathcal{S}_\omega(w_n^k)\right\} = 0, \quad (6.57)$$

$$\lim_{n\to\infty}\left\{\mathcal{I}_\omega(\psi_n(0)) - \sum_{j=1}^{k}\mathcal{I}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) - \mathcal{I}_\omega(w_n^k)\right\} = 0, \quad (6.58)$$

$$\lim_{n\to\infty}\left\{\mathcal{K}(\psi_n(0)) - \sum_{j=1}^{k}\mathcal{K}\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) - \mathcal{K}(w_n^k)\right\} = 0. \quad (6.59)$$

Note here that (6.55) together with (6.45) yields

$$\sup_{k\in\mathbb{N}}\limsup_{n\to\infty}\left\|\langle\nabla\rangle e^{it\Delta}w_n^k\right\|_{S(\mathbb{R})} \lesssim \sup_{k\in\mathbb{N}}\limsup_{n\to\infty}\left\|w_n^k\right\|_{H^1} < \infty. \quad (6.60)$$

Next, we define the *nonlinear profile*. Let U_n^j be the solution to (NLS) with $U_n^j(0) = \frac{\langle\nabla\rangle^{-1}\langle\lambda_n^j\nabla\rangle}{\lambda_n^j}e^{-it_n^j\Delta}g_n^j\widetilde{u}^j$, so that $e^{it\Delta}U_n^j(0) = G_n^j(\sigma_n^j e^{it\Delta}\widetilde{u}^j)$ (see (6.52)). Thus, U_n^j satisfies

$$U_n^j(t) = G_n^j(\sigma_n^j e^{it\Delta}\widetilde{u}^j) + i\int_0^t e^{i(t-t')\Delta}\left\{ f(U_n^j) + |U_n^j|^{2^*-2}U_n^j \right\}(t')\,dt'. \quad (6.61)$$

Undoing the transformations G_n^j and σ_n^j in (6.61), we have the equation

$$\begin{aligned}\widetilde{\psi}_n^j(t) &= e^{it\Delta}\widetilde{u}^j \\ &\quad + i\int_{-\frac{t_n^j}{(\lambda_n^j)^2}}^{t} e^{i(t-t')\Delta}(\sigma_n^j)^{-1}\left\{(\lambda_n^j)^{\frac{d+2}{2}}f\left((\lambda_n^j)^{-\frac{d-2}{2}}\sigma_n^j\widetilde{\psi}_n^j\right) + |\sigma_n^j\widetilde{\psi}_n^j|^{2^*-2}\sigma_n^j\widetilde{\psi}_n^j\right\}(t')\,dt',\end{aligned} \quad (6.62)$$

where

$$\widetilde{\psi}_n^j := (\sigma_n^j)^{-1}(G_n^j)^{-1}U_n^j. \quad (6.63)$$

We define the nonlinear profile $\widetilde{\psi}^j$ as a solution to the limit equation of (6.62):

$$\begin{aligned}\widetilde{\psi}^j(t) &= e^{it\Delta}\widetilde{u}^j \\ &\quad + i\int_{-\frac{t_\infty^j}{(\lambda_\infty^j)^2}}^{t} e^{i(t-t')\Delta}(\sigma_\infty^j)^{-1}\left\{(\lambda_\infty^j)^{\frac{d+2}{2}}f\left((\lambda_\infty^j)^{-\frac{d-2}{2}}\sigma_\infty^j\widetilde{\psi}^j\right) + |\sigma_\infty^j\widetilde{\psi}^j|^{2^*-1}\sigma_\infty^j\widetilde{\psi}^j\right\}(t')\,dt' \\ &= e^{it\Delta}\widetilde{u}^j + i\int_{-\frac{t_\infty^j}{(\lambda_\infty^j)^2}}^{t} e^{i(t-t')\Delta}(\sigma_\infty^j)^{-1}\mathcal{N}_j(\sigma_\infty^j\widetilde{\psi}^j(t'))\,dt',\end{aligned} \quad (6.64)$$

where

$$\sigma_\infty^j := \begin{cases} 1 & \text{if } \lambda_\infty^j = 1, \\ |\nabla|^{-1}\langle\nabla\rangle & \text{if } \lambda_\infty^j = 0 \end{cases} \tag{6.65}$$

and

$$\mathcal{N}_j(u) := \begin{cases} f(u) + |u|^{2^*-2}u & \text{if } \lambda_\infty^j = 1, \\ |u|^{2^*-2}u & \text{if } \lambda_\infty^j = 0. \end{cases} \tag{6.66}$$

When $-\dfrac{t_\infty^j}{(\lambda_\infty^j)^2} \in \{\pm\infty\}$, we regard (6.64) as the final value problem at $\pm\infty$.

Let $I^j := (T_{\min}^j, T_{\max}^j)$ be the maximal interval where the nonlinear profile $\widetilde{\psi}^j$ exists. Note that $T_{\min}^j = -\infty$ when $t_\infty^j/(\lambda_\infty^j)^2 = +\infty$, and $T_{\max}^j = +\infty$ when $t_\infty^j/(\lambda_\infty^j)^2 = -\infty$. We see from the construction of the nonlinear profile that $\widetilde{\psi}^j \in C(I^j, H^1(\mathbb{R}^d))$ and

$$\lim_{n\to\infty} \left\| \widetilde{\psi}^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right) - e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j \right\|_{H^1} = 0. \tag{6.67}$$

Lemma 6.5. *There exists $\delta > 0$ with the following property: Let $j \in \mathbb{N}$ and assume that*

$$\|\widetilde{u}^j\|_{H^1} \le \delta. \tag{6.68}$$

Then, we have $I^j = \mathbb{R}$ and

$$\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} \lesssim \|\widetilde{u}^j\|_{H^1}. \tag{6.69}$$

Proof of Lemma 6.5. This lemma follows from the standard small-data well-posedness theory. □

Moreover, we can verify the following lemma in a way similar to Lemma 4.2:

Lemma 6.6. *Assume $d \ge 3$. Let I be an interval, $A, B > 0$, and let u be a function such that*

$$\|u\|_{L^\infty(I,H^1)} \le A, \qquad \|u\|_{W_{p_1}(I)\cap W(I)} \le B. \tag{6.70}$$

Let $\varepsilon > 0$ and suppose that

$$\left\|\langle\nabla\rangle\left(i\frac{\partial}{\partial t}u + \Delta u + \sigma_\infty^{-1}\mathcal{N}_j(\sigma_\infty^j u)\right)\right\|_{L^{\frac{2(d+2)}{d+4}}(I,L^{\frac{2(d+2)}{d+4}})} \le \varepsilon. \tag{6.71}$$

Then, we have

$$\|\langle\nabla\rangle u\|_{S(I)} \lesssim C(A,B) + \varepsilon, \tag{6.72}$$

where $C(A,B)$ is some constant depending on A and B.

Lemma 6.7 (Properties of nonlinear profiles)**.** *Let $\widetilde{\psi}^{j_1}$ be a nonlinear profile, and suppose that it is non-trivial. Then, we have*

$$\sigma_\infty^{j_1}\widetilde{\psi}^{j_1}(t) \in \begin{cases} A_{\omega,+} & \textit{if} \quad \lambda_\infty^{j_1} = 1, \\ A_0 & \textit{if} \quad \lambda_\infty^{j_1} = 0 \end{cases} \qquad \textit{for any } t \in I^{j_1}. \tag{6.73}$$

Proof of Lemma 6.7. Note first that when $\widetilde{\psi}^{j_1}$ is non-trivial, we see from (6.67) that the corresponding linear profile $\widetilde{u}^{j_1}$ is also non-trivial.

We shall show that

$$\mathcal{I}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) < \frac{m_\omega + m_\omega^*}{2} \quad \text{for any } j \text{ and sufficiently large } n \in \mathbb{N}, \tag{6.74}$$

$$\mathcal{K}\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) > 0 \quad \text{for any } j \geq 1 \text{ and sufficiently large } n \in \mathbb{N}, \tag{6.75}$$

$$\mathcal{S}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) < \frac{m_\omega + m_\omega^*}{2} \quad \text{for any } j \text{ and sufficiently large } n \in \mathbb{N}. \tag{6.76}$$

It follows from (6.43), $\mathcal{K}(\psi_n(0)) > 0$ and (6.58) that

$$\begin{aligned} m_\omega^* + o_n(1) &= \mathcal{S}_\omega(\psi_n(0)) \\ &\geq \mathcal{I}_\omega(\psi_n(0)) = \sum_{j=1}^{k} \mathcal{I}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) + \mathcal{I}_\omega(w_n^k) + o_n(1). \end{aligned} \tag{6.77}$$

Hence, (6.77) together with $m_\omega^* < m_\omega$ and the positivity of $\mathcal{I}_\omega$ shows (6.74). Moreover, (6.74) together with the definition of $\widetilde{m}_\omega$ (see (1.12)) shows (6.75).

Note here that

$$\liminf_{n\to\infty}\left\|\nabla g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right\|_{L^2} = \left\|\nabla\sigma_\infty^j \widetilde{u}^j\right\|_{L^2} > 0, \quad \text{provided that } \widetilde{u}^j \text{ is non-trivial.} \tag{6.78}$$

Hence, we see from Lemma 6.1 together with (6.78) that

$$\mathcal{S}_\omega\left(g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right) \geq 0 \qquad \text{for any } j \geq 1 \text{ and sufficiently large } n \in \mathbb{N}. \tag{6.79}$$

Thus, (6.57) together with (6.43) and (6.79) gives us (6.76).

We shall prove (6.73). Suppose that $\lambda_\infty^{j_1} = 1$, so that $\sigma_\infty^j = 1$. Then, it follows from (6.67) that

$$\lim_{n\to\infty}\left\|(\sigma_\infty^j \widetilde{\psi}^{j_1})\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right) - g_n^{j_1}\left(\sigma_n^{j_1} e^{-i\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\Delta}\widetilde{u}^{j_1}\right)\right\|_{H^1} = 0. \tag{6.80}$$

This together with (6.74) gives us that

$$\lim_{n\to\infty}\mathcal{I}_\omega\left(\sigma_\infty^{j_1}\widetilde{\psi}^{j_1}\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) \leq \frac{m_\omega + m_\omega^*}{2} < \widetilde{m}_\omega, \tag{6.81}$$

which together with the definition of $\widetilde{m}_\omega$ shows

$$\mathcal{K}\left(\sigma_\infty^{j_1}\widetilde{\psi}^{j_1}\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) > 0 \qquad \text{for any sufficiently large } n \in \mathbb{N}. \tag{6.82}$$

Moreover, (6.76) together with (6.80) yields that

$$\mathcal{S}_\omega\left(\sigma_\infty^{j_1}\widetilde{\psi}^{j_1}\Big(-\frac{t_n^j}{(\lambda_n^j)^2}\Big)\right) < m_\omega \qquad \text{for any sufficiently large } n \in \mathbb{N}. \tag{6.83}$$

Hence, $\sigma_\infty^{j_1}\widetilde{\psi}^{j_1}\Big(-\frac{t_n^j}{(\lambda_n^j)^2}\Big) \in A_{\omega,+}$ for any sufficiently large $n \in \mathbb{N}$. Then, (6.73) immediately follows from Lemma 6.4.

On the other hand, suppose $\lambda_\infty^{j_1} = 0$. Then, we see from (6.76) that

$$\frac{m_\omega + m_\omega^*}{2} > \frac{1}{2}\left\|\nabla g_n^{j_1}\left(\sigma_n^{j_1} e^{-i\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\Delta}\widetilde{u}^{j_1}\right)\right\|_{L^2}^2 - \frac{1}{2^*}\left\|g_n^{j_1}\left(\sigma_n^{j_1} e^{-i\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\Delta}\widetilde{u}^{j_1}\right)\right\|_{L^{2^*}}^{2^*}, \tag{6.84}$$

which together with Lemma 1.2 and (6.67) shows

$$\frac{1}{d}\sigma^{\frac{d}{2}} > \mathcal{H}_0\left(\sigma_\infty^{j}\widetilde{\psi}^{j_1}\left(-\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\right)\right). \tag{6.85}$$

Moreover, (6.84) together with (6.75) yields that

$$\frac{m_\omega + m_\omega^*}{2} > \frac{1}{d}\left\|\nabla g_n^{j_1}\left(\sigma_n^{j_1} e^{-i\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\Delta}\widetilde{u}^{j_1}\right)\right\|_{L^2}^2. \tag{6.86}$$

Hence, we see from (6.67) that

$$\sigma^{\frac{d}{2}} > \left\|\nabla\sigma_\infty^{j}\widetilde{\psi}^{j_1}\left(-\frac{t_n^{j_1}}{(\lambda_n^{j_1})^2}\right)\right\|_{L^2}^2. \tag{6.87}$$

Thus, we have shown (6.73). $\square$

Lemma 6.8. *There exists $j_0 \in \mathbb{N}$ such that $I^j = \mathbb{R}$ for any $j > j_0$, where I^j denotes the maximal interval where the nonlinear profile $\widetilde{\psi}^j$ exists, and*

$$\sum_{j>j_0}\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^2 \lesssim \sum_{j>j_0}\left\|\widetilde{u}^j\right\|_{H^1}^2 < \infty. \tag{6.88}$$

Proof of Lemma 6.8. It follows from (6.55) together with (6.45) that

$$\sum_{j=1}^{\infty}\lim_{n\to\infty}\left\||\nabla|^s g_n^j\left(\sigma_n^j e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\widetilde{u}^j\right)\right\|_{L^2}^2 \lesssim m_\omega + \frac{m_\omega}{\omega} \qquad \text{for } s = 0, 1. \tag{6.89}$$

We see from (6.89) that

$$\sum_{j=1}^{\infty}\left\|\widetilde{u}^j\right\|_{H^1}^2 < \infty \tag{6.90}$$

and therefore

$$\lim_{j\to\infty}\left\|\widetilde{u}^j\right\|_{H^1} = 0. \tag{6.91}$$

Hence, Lemma 6.5 together with (6.67) and (6.91) shows the desired result. $\square$

Now, for any $j \in \mathbb{N}$, we define the space W^j by

$$W^j := \begin{cases} W_{p_1} \cap W & \text{if } \ \lambda_\infty^j = 1, \\ \quad W & \text{if } \ \lambda_\infty^j = 0. \end{cases} \tag{6.92}$$

Put

$$\psi_n^j := G_n^j \sigma_n^j \widetilde{\psi}^j. \tag{6.93}$$

The maximal interval where ψ_n^j exists is $I_n^j := \Big((\lambda_n^j)^2 T_{\min}^j + t_n^j, (\lambda_n^j)^2 T_{\max}^j + t_n^j\Big)$.

Lemma 6.9. *Let $k \in \mathbb{N}$ and assume that*

$$\left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W^j(I^j)} < \infty \qquad \text{for any } 1 \le j \le k, \tag{6.94}$$

where I^j denotes the maximal interval where $\widetilde{\psi}^j$ exists. Then, we have $I^j = \infty$,

$$\|\nabla \psi_n^j\|_{W_{1+\frac{4}{d}}(\mathbb{R})} + \|\psi_n^j\|_{W_{p_1}(\mathbb{R})\cap W(\mathbb{R})} \lesssim \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} \lesssim 1 \qquad \text{for any } 1 \le j \le k, \tag{6.95}$$

and there exists $B > 0$ with the following property: For any $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$\sup_{n \ge N_k} \left(\|\nabla \psi_n^j\|_{W_{1+\frac{4}{d}}(\mathbb{R})} + \left\| \sum_{j=1}^k \psi_n^j \right\|_{W_{p_1}(\mathbb{R})\cap W(\mathbb{R})} \right) \le B. \tag{6.96}$$

Furthermore, if (6.94) holds for any k, then we have

$$\lim_{k\to\infty}\lim_{n\to\infty} \left\| \psi_n^j |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} = 0 \quad \text{for any } j \ge 1 \text{ and } p_1 \le p \le 2^* - 1. \tag{6.97}$$

Proof of Lemma 6.9. Assume that $\lambda_\infty^j = 0$. Then, $\sigma_\infty^j \widetilde{\psi}^j$ is a solution to (NLS_0) (see (6.64)). Since I^j coincides with the maximal interval where $\sigma_\infty^j \widetilde{\psi}^j$ exists, Theorem 6.1 together with Lemma 6.7 shows $I^j = \mathbb{R}$. On the other hand, when $\lambda_\infty^j = 1$, $\sigma_\infty^j \widetilde{\psi}^j$ is a solution to (NLS) and therefore Theorem 4.1 (iv) together with the hypothesis (6.94) shows $I^j = \mathbb{R}$.

We shall show that

$$\|\nabla \psi_n^j\|_{W_{1+\frac{4}{d}}(\mathbb{R})} + \|\psi_n^j\|_{W_{p_1}(\mathbb{R})\cap W(\mathbb{R})} \lesssim \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}. \tag{6.98}$$

The Mihlin multiplier theorem gives us that

$$\begin{aligned} \|\psi_n^j\|_{W(\mathbb{R})} &= \left\|\sigma_n^j \widetilde{\psi}^j\right\|_{W(\mathbb{R})} = \left\|\sigma_n^j (\sigma_\infty^j)^{-1} \sigma_\infty^j \widetilde{\psi}^j\right\|_{W(\mathbb{R})} \\ &\lesssim \left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W(\mathbb{R})} \lesssim \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} \qquad \text{for any } j \ge 1 \text{ and } n \ge 1. \end{aligned} \tag{6.99}$$

Similarly, we have

$$\|\nabla \psi_n^j\|_{W_{1+\frac{4}{d}}(\mathbb{R})} \lesssim \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} \qquad \text{for any } j \ge 1 \text{ and } n \ge 1. \tag{6.100}$$

Moreover, using the Hölder inequality, the Mihlin multiplier theorem and (6.99), we obtain

$$
\begin{aligned}
\|\psi_n^j\|_{W_{p_1}(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}} &\le \left\|G_n^j\sigma_n^j\widetilde{\psi}^j\right\|_{W_{1+\frac{4}{d}}(\mathbb{R})}^{(d+2)\left\{1-\frac{(d-2)(p_1-1)}{4}\right\}} \left\|G_n^j\sigma_n^j\widetilde{\psi}^j\right\|_{W(\mathbb{R})}^{\frac{(d+2)\{d(p_1-1)-4\}}{4}} \\
&= \left\|\lambda_n^j\sigma_n^j\widetilde{\psi}^j\right\|_{W_{1+\frac{4}{d}}(\mathbb{R})}^{(d+2)\left\{1-\frac{(d-2)(p_1-1)}{4}\right\}} \left\|\sigma_n^j\widetilde{\psi}^j\right\|_{W(\mathbb{R})}^{\frac{(d+2)\{d(p_1-1)-4\}}{4}} \\
&\lesssim \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}}.
\end{aligned}
\tag{6.101}
$$

We shall show (6.95). In view of (6.98), it suffices to prove that

$$
\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} \lesssim 1 \qquad \text{for any } 1\le j\le k. \tag{6.102}
$$

When $\lambda_\infty^j = 0$, it follows from the Strichartz estimate and (4.8) that

$$
\begin{aligned}
\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} &\lesssim \|\widetilde{u}^j\|_{H^1} + \left\|\nabla\left\{\,|\sigma_\infty^j\widetilde{\psi}^j|^{2^*-2}(\sigma_\infty^j\widetilde{\psi}^j)\,\right\}\right\|_{L^2(\mathbb{R},L^{\frac{2d}{d+2}})} \\
&\lesssim \|\widetilde{u}^j\|_{H^1} + \left\|\nabla\sigma_\infty^j\widetilde{\psi}^j\right\|_{V(\mathbb{R})}\left\|\sigma_\infty^j\widetilde{\psi}^j\right\|_{W(\mathbb{R})}^{2^*-2} \\
&\le \|\widetilde{u}^j\|_{H^1} + \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{L^\infty(\mathbb{R},L^2)}^{\frac{4}{d+2}}\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{L^2(\mathbb{R},L^{2^*})}^{\frac{d-2}{d+2}}\left\|\sigma_\infty^j\widetilde{\psi}^j\right\|_{W(\mathbb{R})}^{2^*-2}.
\end{aligned}
\tag{6.103}
$$

Here, (6.40) together with Lemma 6.7 shows that

$$
\left\|\nabla\sigma_\infty^j\widetilde{\psi}^j\right\|_{L^\infty(\mathbb{R},L^2)} = \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{L^\infty(\mathbb{R},L^2)} < \infty. \tag{6.104}
$$

Hence, (6.103) together with (6.104) and the hypothesis (6.94) shows

$$
\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} < \infty \qquad \text{for any } 1\le j\le k \text{ with } \lambda_\infty^j = 0. \tag{6.105}
$$

When $\lambda_\infty^j = 1$, Lemma 4.2 together with Lemma 6.7 and (6.94) shows

$$
\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})} < \infty \qquad \text{for any } 1\le j\le k \text{ with } \lambda_\infty^j = 1. \tag{6.106}
$$

Suppose here that $k\le j_0$; j_0 is the number found in Lemma 6.8. Then, (6.105) and (6.106) implies (6.102). On the other hand, when $k > j_0$, we see from Lemma 6.8 that

$$
\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^2 \lesssim \sum_{1\le j\le j_0}\left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^2 + \sum_{j>j_0}\|\widetilde{u}^j\|_{H^1}^2 < \infty. \tag{6.107}
$$

Thus, we have shown (6.102).

We shall prove (6.96). It is sufficient to consider the case $k\ge j_0$. Using the elementary inequality

$$
\left|\sum_{1\le j\le k}\psi_n^j\right|^q - \sum_{1\le j\le k}|\psi_n^j|^q \le C_{k,q}\sum_{1\le j\le k}\sum_{\substack{1\le j'\le k;\\ j'\neq j}}|\psi_n^j|^{q-1}\left|\psi_n^{j'}\right|, \quad 1<q<\infty, \tag{6.108}
$$

where $C_{k,q}$ is some constant depends only on k and q, we have

$$\left\| \sum_{1\le j\le k} \psi_n^j \right\|_{W(\mathbb{R})}^{\frac{2(d+2)}{d-2}} \le \sum_{1\le j\le k} \|\psi_n^j\|_{W(\mathbb{R})}^{\frac{2(d+2)}{d-2}} + C_k \sum_{1\le j\le k} \sum_{\substack{1\le j'\le k;\\ j'\neq j}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi_n^j\right|^{\frac{d+6}{d-2}} \left|\psi_n^{j'}\right|, \tag{6.109}$$

where $C_k > 0$ is some constant depending only on k and d. We see from Lemma 6.8 and (6.95) that

$$\begin{aligned} \sum_{1\le j\le k} \|\psi_n^j\|_{W(\mathbb{R})}^{\frac{2(d+2)}{d-2}} &\lesssim \sum_{1\le j\le j_0} \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^{\frac{2(d+2)}{d-2}} + \sum_{j>j_0} \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^{\frac{2(d+2)}{d-2}} \\ &\lesssim \sum_{1\le j\le j_0} \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^{\frac{2(d+2)}{d-2}} + \sum_{j>j_0} \|\widetilde{w}^j\|_{H^1}^2 < \infty. \end{aligned} \tag{6.110}$$

Moreover, we see from (6.152) that there exists $N_{j,k} \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi_n^j\right|^{\frac{d+6}{d-2}} \left|\psi_n^{j'}\right| \le \frac{1}{C_k k^2} \qquad \text{for any } n \ge N_{j,k}. \tag{6.111}$$

Combining (6.109) with (6.110) and (6.111), we can take $B_0 > 0$ with the property that for any $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$\sup_{n\ge N_k} \left\| \sum_{j=1}^{k} \psi_n^j \right\|_{W(\mathbb{R})} \le B_0. \tag{6.112}$$

Next, we consider the estimate in $W_{p_1}(\mathbb{R})$. Using the elementary inequality (6.108) again, we have

$$\begin{aligned} \left\| \sum_{1\le j\le k} \psi_n^j \right\|_{W_{p_1}(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}} &\le \sum_{1\le j\le k} \|\psi_n^j\|_{W_{p_1}(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}} \\ &\quad + C_k' \sum_{1\le j\le k} \sum_{\substack{1\le j'\le k;\\ j'\neq j}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi_n^j\right|^{\frac{(d+2)(p_1-1)-2}{2}} \left|\psi_n^{j'}\right|, \end{aligned} \tag{6.113}$$

where C_k' is some constant depending only on k, d and p_1. We see from (6.95) and Lemma 6.8 that

$$\sum_{1\le j\le k} \|\psi_n^j\|_{W_{p_1}(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}} \lesssim \sum_{1\le j\le j_0} \left\|\langle\nabla\rangle\widetilde{\psi}^j\right\|_{S(\mathbb{R})}^{\frac{(d+2)(p_1-1)}{2}} + \sum_{j>j_0} \|\widetilde{w}^j\|_{H^1}^2 < \infty. \tag{6.114}$$

Next, we consider the second term on the right-hand side of (6.113). Using the condition (6.152), we can take $N_{j,k}' \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|\psi_n^j\right|^{\frac{(d+2)(p_1-1)-2}{2}} \left|\psi_n^{j'}\right| \le \frac{1}{C_k' k^2} \qquad \text{for any } n \ge N_{j,k}'. \tag{6.115}$$

Combining (6.113) with (6.114) and (6.115), we can take B_1 with the property that for any $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$\sup_{n \geq N_k} \left\| \sum_{j=1}^{k} \psi_n^j \right\|_{W_{p_1}(\mathbb{R})} \leq B_1. \tag{6.116}$$

Similarly, we have

$$\sup_{n \geq N_k} \left\| \sum_{j=1}^{k} \nabla \psi_n^j \right\|_{W_{1+\frac{4}{d}}(\mathbb{R})} \leq B_1. \tag{6.117}$$

Thus, we have proved (6.96).

Finally, we shall prove (6.97). For each $1 \leq j \leq k$, let $\{v_m^j\}$ be a sequence in $C_c(\mathbb{R}^d \times \mathbb{R})$ such that

$$\lim_{m \to \infty} \left\| \langle \nabla \rangle \left(\widetilde{\psi}^j - v_m^j \right) \right\|_{V_p(\mathbb{R})} = 0. \tag{6.118}$$

Then, using the Hölder inequality, the Strichartz estimate and (6.60), we have

$$\begin{aligned}
&\left\| \psi_n^j |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\
&= (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \left(\sigma_n^j \widetilde{\psi}^j \right) (G_n^j)^{-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\
&\leq (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \sigma_n^j \left(\widetilde{\psi}^j - v_m^j \right) \right\|_{W_p(\mathbb{R})} \left\| (G_n^j)^{-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&\quad + (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \left(\sigma_n^j v_m^j \right) (G_n^j)^{-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}.
\end{aligned} \tag{6.119}$$

Using the Strichartz estimate, the Mihlin multiplier theorem and (6.118), we estimate the first term on the right-hand side of (6.119) as follows:

$$\begin{aligned}
&(\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \sigma_n^j \left(\widetilde{\psi}^j - v_m^j \right) \right\|_{W_p(\mathbb{R})} \left\| (G_n^j)^{-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&= (\lambda_n^j)^{\frac{4-(d-2)(p-1)}{2(p-1)}} \left\| \sigma_n^j \left(\widetilde{\psi}^j - v_m^j \right) \right\|_{W_p(\mathbb{R})} \left\| |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&\lesssim (\lambda_n^j)^{\frac{4-(d-2)(p-1)}{2(p-1)}} \left\| \langle \nabla \rangle \left(\widetilde{\psi}^j - v_m^j \right) \right\|_{V_p(\mathbb{R})} \left\| w_n^k \right\|_{H^1} = o_m(1).
\end{aligned} \tag{6.120}$$

We consider the second term on the right-hand side of (6.119). Note here that $\frac{2(d+2)(p-1)}{d(p-1)+4} \leq \frac{d+2}{d-1} \leq 2$ for $d \geq 4$. When $s = 0$, we have by the Hölder inequality and (6.49) that

$$\begin{aligned}
&(\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \left(\sigma_n^j v_m^j \right) (G_n^j)^{-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\
&\leq \lambda_n^j \left\| \sigma_n^j v_m^j \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{2(p-1)+4}}} \left\| e^{it\Delta} w_n^k \right\|_{W(\mathbb{R})} \to 0 \qquad \text{as } n \to \infty \text{ and } k \to \infty.
\end{aligned} \tag{6.121}$$

Next, we consider the case where $s=1$. Let $K_m^j := \{|x| \le R_m^j\} \times [-T_m^j, T_m^j]$ be the rectangle containing the support of v_m^j. Using the Hölder inequality and employing Lemma 2.5 in [18] (see also [17]), we have [2]

$$
\begin{aligned}
&(\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \left(\sigma_n^j v_m^j\right) (G_n^j)^{-1} \nabla e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\
&\le (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)+1} \left\| \sigma_n^j v_m^j \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{4-2(p-1)}}} \left\| \nabla e^{it\Delta} (g_n^j)^{-1} w_n^k \right\|_{L_{t,x}^2(K_m^j)} \\
&\lesssim (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)+1} \left\| \sigma_n^j v_m^j \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{4-2(p-1)}}} (T_m^j)^{\frac{2}{d+2}} (R_m^j)^{\frac{3d+2}{2(d+2)}} \left\| e^{it\Delta} w_n^k \right\|_{W(\mathbb{R})}^{\frac{1}{3}} \left\| \nabla w_n^k \right\|_{L^2}^{\frac{2}{3}}.
\end{aligned}
\tag{6.123}
$$

Hence, we see from (6.49) that

$$
\lim_{k\to\infty} \lim_{n\to\infty} (\lambda_n^j)^{\frac{d(p-1)+4}{2(p-1)}-(d-2)} \left\| \left(\sigma_n^j v_m^j\right) (G_n^j)^{-1} \nabla e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}(K_m^j)} = 0. \tag{6.124}
$$

Combining (6.119) with (6.120), (6.121) and (6.124), we obtain (6.97). ☐

Lemma 6.10 (cf. Lemma 5.6 in [13])**.** *Let j_0 be the number found in Lemma 6.8. Then, there exists $1 \le j \le j_0$ such that*

$$
\left\| \sigma_\infty^j \widetilde{\psi}^j \right\|_{W^j(I^j)} = \infty, \tag{6.125}
$$

where I^j denotes the maximal interval where the nonlinear profile $\widetilde{\psi}^j$ exists.

Proof of Lemma 6.10. We see from Lemma 6.8 that

$$
\left\| \sigma_\infty^j \widetilde{\psi}^j \right\|_{W^j(I^j)} \lesssim \left\| \langle \nabla \rangle \widetilde{\psi}^j \right\|_{S(\mathbb{R})} < \infty \quad \text{for any } j > j_0. \tag{6.126}
$$

Suppose the contrary that

$$
\left\| \sigma_\infty^j \widetilde{\psi}^j \right\|_{W^j(I^j)} < \infty \qquad \text{for any } 1 \le j \le j_0, \tag{6.127}
$$

so that

$$
\left\| \sigma_\infty^j \widetilde{\psi}^j \right\|_{W^j(I^j)} < \infty \qquad \text{for any } j \ge 1. \tag{6.128}
$$

Let k be a sufficiently large number to be specified later. Then, we define the function $\psi_n^{k\text{-}app}$ by

$$
\psi_n^{k\text{-}app}(x,t) := \sum_{j=1}^{k} \psi_n^j(x,t) + e^{it\Delta} w_n^k(x). \tag{6.129}
$$

[2] When $d=3$ and $2 < \frac{2(d+2)(p-1)}{d(p-1)+4}$ (hence $p>3$), we estimate as follows:

$$
\begin{aligned}
&\left\| \left(\sigma_n^j v_m^j\right) (G_n^j)^{-1} \nabla e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\
&\le \lambda_n^j \left\| \sigma_n^j v_m^j \right\|_{L_{t,x}^\infty} \left\| \nabla e^{it\Delta} (g_n^j)^{-1} w_n^k \right\|_{L_{t,x}^2(K_m^j)}^{\frac{2}{p-1}} \left\| \nabla e^{it\Delta} (g_n^j)^{-1} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{p-3}{p-1}}.
\end{aligned}
\tag{6.122}
$$

It follows from Lemma 6.9 together with the hypothesis (6.128) that ψ_n^j exists globally in time for any $j \geq 1$ and hence so does $\psi_n^{k\text{-}app}$.

We see from Lemma 6.9 and (6.60) that there exists $B > 0$ with the following property: there exists $N_{1,k} \in \mathbb{N}$ such that

$$\sup_{n \geq N_{1,k}} \left\| \psi_n^{k\text{-}app} \right\|_{W_{p_1}(\mathbb{R}) \cap W(\mathbb{R})} \leq B. \tag{6.130}$$

Moreover, it follows from (6.52) that

$$\begin{aligned} \left\| \psi_n(0) - \psi_n^{k\text{-}app}(0) \right\|_{H^1} &= \left\| \psi_n(0) - \sum_{j=1}^{k} \psi_n^j(0) - w_n^k \right\|_{H^1} \\ &\lesssim \sum_{j=1}^{k} \left\| e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta} \widetilde{u}^j - \widetilde{\psi}^j \left(-\frac{t_n^j}{(\lambda_n^j)^2} \right) \right\|_{H^1}. \end{aligned} \tag{6.131}$$

Hence, (6.67) shows that for any $k \in \mathbb{N}$, there exists $N_{2,k} \in \mathbb{N}$ such that

$$\sup_{n \geq N_{2,k}} \left\| \psi_n(0) - \psi_n^{k\text{-}app}(0) \right\|_{H^1} \leq 1. \tag{6.132}$$

Now, let δ be the constant found in Proposition 5.6 which is determined by the bound (6.45) and B.

We see from the estimate (6.131) together with the Strichartz estimate and (6.67) that for any $k \in \mathbb{N}$, there exists $N_{3,k} \in \mathbb{N}$ such that

$$\sup_{n \geq N_{3,k}} \left\| \langle \nabla \rangle e^{it\Delta} \left(\psi_n(0) - \psi_n^{k\text{-}app}(0) \right) \right\|_{V_{p_1}} < \delta. \tag{6.133}$$

We shall show that there exist $k_0 \in \mathbb{N}$ and $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\left\| \langle \nabla \rangle \left(i \frac{\partial \psi_n^{k_0\text{-}app}}{\partial t} + \Delta \psi_n^{k_0\text{-}app} + f(\psi_n^{k_0\text{-}app}) + |\psi_n^{k_0\text{-}app}|^{2^*-2} \psi_n^{k_0\text{-}app} \right) \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &< \delta \qquad \text{for any } n \geq N_0. \end{aligned} \tag{6.134}$$

Before proving this, we remark that (6.134) together with the long-time perturbation theory leads to an absurd conclusion; Indeed, we see from (6.130), (6.132) and (6.133) that

$$\left\| \psi_n(0) - \psi_n^{k_0\text{-}app} \right\|_{H^1} \leq 1, \tag{6.135}$$

$$\left\| \psi_n^{k_0\text{-}app} \right\|_{W_{p_1}(\mathbb{R}) \cap W(\mathbb{R})} \leq B, \tag{6.136}$$

$$\left\| \langle \nabla \rangle e^{it\Delta} \left(\psi_n(0) - \psi_n^{k_0\text{-}app}(0) \right) \right\|_{V_{p_1}(\mathbb{R})} < \delta \tag{6.137}$$

for any $n \geq \max\{N_{1,k_0}, N_{2,k_0}, N_{3,k_0}, N_0\}$. Hence, employing Proposition 5.6 (Long-time perturbation theory), we conclude that

$$\|\psi_n\|_{W_p(\mathbb{R}) \cap W(\mathbb{R})} < \infty \qquad \text{for any } n \geq \max\{N_{1,k_0}, N_{2,k_0}, N_{3,k_0}, N_0\}, \tag{6.138}$$

which contradicts (6.44): hence, Lemma 6.10 holds.

It remains to prove (6.134). Note that

$$
\begin{aligned}
& i\frac{\partial \psi_n^{k\text{-}app}}{\partial t} + \Delta \psi_n^{k\text{-}app} + f(\psi_n^{k\text{-}app}) + |\psi_n^{k\text{-}app}|^{2^*-2}\psi_n^{k\text{-}app} \\
&= \mathcal{N}\big(\psi_n^{k\text{-}app}\big) - \mathcal{N}\big(\psi_n^{k\text{-}app} - e^{it\Delta}w_n^k\big) + \sum_{j=1}^{k} i\frac{\partial \psi_n^j}{\partial t} + \Delta\psi_n^j + \mathcal{N}\Bigg(\sum_{j=1}^{k}\psi_n^j\Bigg),
\end{aligned}
\tag{6.139}
$$

where

$$
\mathcal{N}(u) := f(u) + |u|^{2^*-2}u, \tag{6.140}
$$

$$
\mathcal{N}_j(u) := \begin{cases} f(u) + |u|^{2^*-2}u & \text{if } \lambda_\infty^j = 1, \\ |u|^{2^*-2}u & \text{if } \lambda_\infty^j = 0. \end{cases} \tag{6.141}
$$

Hence, it suffices for (6.134) to show that

$$
\lim_{k\to\infty}\lim_{n\to\infty}\left\| \langle\nabla\rangle \left\{ \mathcal{N}(\psi_n^{k\text{-}app}) - \mathcal{N}\left(\psi_n^{k\text{-}app} - e^{it\Delta}w_n^k\right)\right\}\right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = 0, \tag{6.142}
$$

$$
\lim_{k\to\infty}\lim_{n\to\infty}\left\| \langle\nabla\rangle \left\{ \sum_{j=1}^{k} i\frac{\partial\psi_n^j}{\partial t} + \Delta\psi_n^j + \mathcal{N}\left(\sum_{j=1}^{k}\psi_n^j\right)\right\}\right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = 0. \tag{6.143}
$$

First, we prove (6.142). Using the growth conditions (A5) and (A6), we verify that

$$
\begin{aligned}
& \left\| \langle\nabla\rangle \left\{ \mathcal{N}(\psi_n^{k\text{-}app}) - \mathcal{N}\left(\psi_n^{k\text{-}app} - e^{it\Delta}w_n^k\right)\right\}\right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
& \lesssim \sum_{j=1}^{2} \left\| |\psi_n^{k\text{-}app}|^{p_j-1} e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \left\| |\psi_n^{k\text{-}app}|^{\frac{4}{d-2}} e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
& \quad + \sum_{j=1}^{2} \left\| e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)p_j}{d+4}}}^{p_j} + \left\| e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(2^*-1)}{d+4}}}^{2^*-1} \\
& \quad + \sum_{j=1}^{2} \left\| |\psi_n^{k\text{-}app}|^{p_j-1} \nabla e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \left\| |\psi_n^{k\text{-}app}|^{2^*-2} \nabla e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
& \quad + \sum_{j=1}^{2} \left\| ||e^{it\Delta}w_n^k|^{p_j-1} \nabla \psi_n^{k\text{-}app} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \left\| ||e^{it\Delta}w_n^k|^{2^*-2} \nabla \psi_n^{k\text{-}app} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
& \quad + \sum_{j=1}^{2} \left\| ||e^{it\Delta}w_n^k|^{p_j-1} \nabla e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \left\| ||e^{it\Delta}w_n^k|^{2^*-2} \nabla e^{it\Delta}w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}},
\end{aligned}
\tag{6.144}
$$

where we must add the terms

$$\left\| |\psi_n^{k\text{-}app}|^{p_2-2} e^{it\Delta} w_n^k \nabla \psi_n^{k\text{-}app} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \qquad \text{if } p_2 > 2, \tag{6.145}$$

$$\left\| |\psi_n^{k\text{-}app}|^{\frac{6-d}{d-2}} e^{it\Delta} w_n^k \nabla \psi_n^{k\text{-}app} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \qquad \text{if } d \le 5 \tag{6.146}$$

to the right-hand side of (6.144).

Since $\frac{2(d+2)}{d} < \frac{2(d+2)p}{d+4} \le \frac{2(d+2)}{d-2}$ for $p_1 \le p \le 2^* - 1$, we easily see from the Hölder inequality, (6.49) and (6.53) that the 3rd, 4th, the final and the 2nd final terms on the right-hand side of (6.144) vanish as $k \to \infty$ and $n \to \infty$. Using Lemma 6.9, we also estimate the 7th and 8th terms as follows:

$$\begin{aligned} &\lim_{k\to\infty}\lim_{n\to\infty} \left\| ||e^{it\Delta} w_n^k|^{p-1} \nabla \psi_n^{k\text{-}app} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\le \lim_{k\to\infty}\lim_{n\to\infty} \left\| e^{it\Delta} w_n^k \right\|_{W_p(\mathbb{R})}^{p-1} \left\| \nabla \psi_n^{k\text{-}app} \right\|_{W_{1+\frac{4}{d}}(\mathbb{R})} = 0. \end{aligned} \tag{6.147}$$

We consider the terms of the form

$$\left\| |\psi_n^{k\text{-}app}|^{p-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}, \qquad 1 + \tfrac{4}{d} < p \le 2^* - 1 \text{ and } s = 0, 1, \tag{6.148}$$

which corresponds to the 1st, 2nd, 5th, 6th terms on the right-hand side of (6.144). Using the Hölder inequality, (6.96) and (6.60), we have

$$\begin{aligned} &\left\| |\psi_n^{k\text{-}app}|^{p-1} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\le \left\| \psi_n^{k\text{-}app} \right\|_{W_p(\mathbb{R})}^{\frac{3(p-1)}{4}} \left\| |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{5-p}{4}} \left\| \psi_n^{k\text{-}app} |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}^{\frac{p-1}{4}} \\ &\lesssim \left\| \left(\sum_{j=1}^{k} \psi_n^j \right) |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}^{\frac{p-1}{4}} + \left\| e^{it\Delta} w_n^k |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}^{\frac{p-1}{4}} \\ &\qquad \text{for any sufficiently large } n. \end{aligned} \tag{6.149}$$

We consider the first term on the right-hand side of (6.149). It follows from Lemmata 6.8 and 6.9 that for any $\eta > 0$, there exists $J(\eta) \in \mathbb{N}$ such that

$$\left(\sum_{j > J(\eta)} \left\| \psi_n^j \right\|_{W_{p_1}(\mathbb{R}) \cap W(\mathbb{R})}^2 \right)^{\frac{1}{2}} < \eta. \tag{6.150}$$

Using the triangle inequality and the Hölder inequality, we have

$$\begin{aligned} &\left\| \left(\sum_{j=1}^{k} \psi_n^j \right) |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} \\ &\le \sum_{j \le J(\eta)} \left\| \psi_n^j |\nabla|^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}} + \left\| \sum_{j > J(\eta)} \psi_n^j \right\|_{W_p(\mathbb{R})} \left\| \nabla e^{it\Delta} w_n^k \right\|_{W_{1+\frac{4}{d}}(\mathbb{R})}. \end{aligned} \tag{6.151}$$

Lemma 6.9 shows the first term on the right-hand side of (6.151) vanishes when k tends to ∞ and then n tends to ∞. Moreover, we see from the elementary inequality (6.108), (6.152), (6.60) and (6.150) that

$$\lim_{n\to\infty}\Big\|\sum_{j>J(\eta)}\psi_n^j\Big\|_{W_p(\mathbb{R})}\|\nabla e^{it\Delta}w_n^J\|_{W_{1+\frac{4}{d}}(\mathbb{R})}\lesssim\Big(\sum_{j>J(\eta)}\|\psi_n^j\|_{W_p(\mathbb{R})}\Big)^{\frac{1}{2}}\Big\|w_n^k\Big\|_{H^1}\lesssim\eta. \tag{6.152}$$

Thus, we find that

$$\lim_{k\to\infty}\lim_{n\to\infty}\Big\|\Big(\sum_{j=1}^k\psi_n^j\Big)|\nabla|^s e^{it\Delta}w_n^k\Big\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}^{\frac{p-1}{4}}=0. \tag{6.153}$$

On the other hand, it follows form the Hölder inequality and (6.49) that

$$\begin{aligned}&\Big\|e^{it\Delta}w_n^k|\nabla|^s e^{it\Delta}w_n^k\Big\|_{L_{t,x}^{\frac{2(d+2)(p-1)}{d(p-1)+4}}}\\&\le\Big\|e^{it\Delta}w_n^k\Big\|_{W_{1+\frac{4}{d}}(\mathbb{R})}^{\frac{2}{p-1}-\frac{d-2}{2}}\Big\|e^{it\Delta}w_n^k\Big\|_{W(\mathbb{R})}^{\frac{d}{2}-\frac{2}{p-1}}\Big\||\nabla|^s e^{it\Delta}w_n^k\Big\|_{W_{1+\frac{4}{d}}}\\&\to 0\qquad\text{as } k\to\infty \text{ and } n\to\infty.\end{aligned} \tag{6.154}$$

Combining (6.149) with (6.153) and (6.154), we obtain

$$\lim_{k\to\infty}\lim_{n\to\infty}\Big\||\psi_n^{k\text{-}app}|^{p-1}|\nabla|^s e^{it\Delta}w_n^k\Big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}=0,\quad 1+\tfrac{4}{d}<p\le 2^*-1 \text{ and } s=0,1 \tag{6.155}$$

Hence, we have proved (6.142).

Finally, we prove (6.143). Noting that

$$\begin{aligned}i\frac{\partial\psi_n^j}{\partial t}+\Delta\psi_n^j&=-\frac{1}{(\lambda_n^j)^2}G_n^j\sigma_n^j\big[(\sigma_\infty^j)^{-1}\mathcal{N}_j(\sigma_\infty^j\widetilde{\psi}^j)\big]\\&=\begin{cases}-\mathcal{N}(\psi_n^j) & \text{if}\quad\lambda_\infty^j=1,\\ -\langle\nabla\rangle^{-1}|\nabla|\Big\{|G_n^j\sigma_\infty^j\widetilde{\psi}^j|^{2^*-2}(G_n^j\sigma_\infty^j\widetilde{\psi}^j)\Big\} & \text{if}\quad\lambda_\infty^j=0,\end{cases}\end{aligned} \tag{6.156}$$

we can verify that

$$\begin{aligned}&\Big\|\langle\nabla\rangle\Big\{\sum_{j=1}^k i\frac{\partial\psi_n^j}{\partial t}+\Delta\psi_n^j+\mathcal{N}\Big(\sum_{j=1}^k\psi_n^j\Big)\Big\}\Big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}\\&\le\Big\|\langle\nabla\rangle\Big\{\mathcal{N}\Big(\sum_{j=1}^J\psi_n^j\Big)-\sum_{j=1}^J\mathcal{N}\left(\psi_n^j\right)\Big\}\Big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}+o_n(1).\end{aligned} \tag{6.157}$$

We see from the growth conditions (A5) and (A6) that

$$\begin{aligned} &\left\| |\nabla|^s \left\{ \mathcal{N}\left(\sum_{j=1}^{J} \psi_n^j \right) - \sum_{j=1}^{J} \mathcal{N}\left(\psi_n^j\right) \right\} \right\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\lesssim \sum_{k=1}^{3} \sum_{j=1}^{J} \sum_{\substack{1\le j'\le J \\ j'\neq j}} \left\| \left|\psi_n^{j'}\right|^{p_k-1} |\nabla|^s \psi_n^j \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}}, \end{aligned} \tag{6.158}$$

where $p_3 := 1 + \frac{4}{d-2}$. Suppose here that $\lim_{n\to\infty} \dfrac{\lambda_n^{j'}}{\lambda_n^j} = \infty$. Then, we have

$$\begin{aligned} &\left\| \left|\psi_n^{j'}\right|^{p_k-1} |\nabla|^s \psi_n^j \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &= (\lambda_n^j)^{1-s+\frac{d-2}{2}\left\{\frac{4}{d-2}-(p_k-1)\right\}} \left(\frac{\lambda_n^j}{\lambda_n^{j'}} \right)^{-\frac{(d-2)(p_l-1)}{2}} \\ &\qquad \left\| \left| \widetilde{\psi}^{j'} \left(\frac{\lambda_n^j x - (x_n^{j'} - x_n^j)}{\lambda_n^{j'}}, \frac{(\lambda_n^j)^2 t - (t_n^{j'} - t_n^j)}{(\lambda_n^{j'})^2} \right) \right|^{p_l-1} |\nabla|^s \widetilde{\psi}^j \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\le (\lambda_n^j)^{1-s+\frac{d-2}{2}\left\{\frac{4}{d-2}-(p_l-1)\right\}} \left(\frac{\lambda_n^j}{\lambda_n^{j'}} \right)^{-\frac{(d-2)(p_l-1)}{2}} \left\| \langle\nabla\rangle \widetilde{\psi}^j \right\|_{S(\mathbb{R})}^{p_l} = o_n(1). \end{aligned} \tag{6.159}$$

When $\lim_{n\to\infty} \dfrac{\lambda_n^j}{\lambda_n^{j'}} = \infty$, we instead have

$$\begin{aligned} &\left\| \left|\psi_n^{j'}\right|^{p_l-1} |\nabla|^s \psi_n^j \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &= (\lambda_n^{j'})^{1-s+\frac{d-2}{2}\left\{\frac{4}{d-2}-(p_l-1)\right\}} \left(\frac{\lambda_n^{j'}}{\lambda_n^j} \right)^{\frac{d}{2}} \\ &\qquad \left\| \left(\frac{\lambda_n^{j'} x - (x_n^j - x_n^{j'})}{\lambda_n^j}, \frac{(\lambda_n^{j'})^2 t - (t_n^j - t_n^{j'})}{(\lambda_n^j)^2} \right) \left| \widetilde{\psi}^{j'} \right|^{p_l-1} |\nabla|^s \widetilde{\psi}^j \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\le (\lambda_n^{j'})^{1-s+\frac{d-2}{2}\left\{\frac{4}{d-2}-(p_l-1)\right\}} \left(\frac{\lambda_n^{j'}}{\lambda_n^j} \right)^{\frac{d}{2}} \left\| \langle\nabla\rangle \widetilde{\psi}^j \right\|_{S(\mathbb{R})}^{p_l} = o_n(1). \end{aligned} \tag{6.160}$$

On the other hand, if $\lim_{n\to\infty} \left\{ \dfrac{\lambda_n^j}{\lambda_n^{j'}} + \dfrac{\lambda_n^j}{\lambda_n^{j'}} \right\} < \infty$, then we deduce from the dichotomy condition (6.152) that

$$\left\| \left|\psi_n^{j'}\right|^{p_l-1} |\nabla|^s \psi_n^j \right\|_{L^2(I, L^{\frac{2d}{d+2}})} = o_n(1). \tag{6.161}$$

Thus, we have shown (6.134). □

Proposition 6.11. *Assume $d \geq 5$ and suppose that $m_\omega^* < m_\omega$. Then, there exists a global solution $\Psi \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ to* (NLS) *such that*

$$\Psi(t) \in A_{\omega,+} \qquad \textit{for any } t \in \mathbb{R}, \tag{6.162}$$

$$\mathcal{S}_\omega\big(\Psi(t)\big) = m_\omega^* \qquad \textit{for any } t \in \mathbb{R}, \tag{6.163}$$

$$\|\Psi\|_{W_p(\mathbb{R}) \cap W(\mathbb{R})} = \infty. \tag{6.164}$$

The function Ψ in Proposition 6.11 is called the critical element. We further give an important properties of the critical element:

Proposition 6.12. *Let Ψ be the solution found in Proposition 6.11. Then, for any $\varepsilon > 0$, there exist $R_\varepsilon > 0$ and $\gamma_\varepsilon \in C([0,\infty), \mathbb{R}^d)$ with $\gamma_\varepsilon(0) = 0$ such that*

$$\int_{|x-\gamma_\varepsilon(t)| \leq R_\varepsilon} |\Psi(x,t)|^2 + |\nabla\Psi(x,t)|^2 \, dx \geq (1-\varepsilon)\, \|\Psi(t)\|_{H^1}^2 \qquad \textit{for any } t \in [0,\infty). \tag{6.165}$$

Furthermore, the momentum of Ψ is zero:

$$\Im \int_{\mathbb{R}^d} \overline{\Psi(x,t)} \nabla\Psi(x,t)\, dx = 0 \qquad \textit{for any } t \in \mathbb{R}. \tag{6.166}$$

We can prove Proposition 6.12 in a way similar to [3, 10]. Hence, we omit it.

Now, we give a proof of Proposition 6.11:

Proof of Proposition 6.11. Using Lemmata 6.8 and 6.10 and reordering indices, we can take a number $J \leq j_1$ such that

$$\begin{aligned} &\left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W_j(I^j)} = \infty \qquad \text{for any } 1 \leq j \leq J, \\ &\left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W_j(\mathbb{R})} < \infty \qquad \text{for any } j > J. \end{aligned} \tag{6.167}$$

We see from Lemma 6.7 that

$$\sigma_\infty^j \widetilde{\psi}^j(t) \in \begin{cases} A_{\omega,+} & \text{if} \quad \lambda_\infty^j = 1, \\ A_0 & \text{if} \quad \lambda_\infty^j = 0 \end{cases} \qquad \text{for any } 1 \leq j \leq J \text{ and } t \in I^j. \tag{6.168}$$

Since $\sigma_\infty^j \widetilde{\psi}^j$ is a solution to (NLS$_0$) when $\lambda_\infty^j = 0$, Theorem 6.1 together with (6.168) shows that

$$\left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W(\mathbb{R})} = \left\|\sigma_\infty^j \widetilde{\psi}^j\right\|_{W_j(I^j)} < \infty, \qquad \text{if } \lambda_\infty^j = 0. \tag{6.169}$$

Hence, we find that

$$\lambda_\infty^j = 1, \quad \sigma_\infty^j = 1 \qquad \text{for any } 1 \leq j \leq J. \tag{6.170}$$

Then, we also have by (6.55), (6.57)–(6.59) that

$$\lim_{n\to\infty}\left\{\||\nabla|^s\psi_n(0)\|_{L^2}^2-\sum_{j=1}^{k}\||\nabla|^s\widetilde{u}^j\|_{L^2}^2-\left\||\nabla|^s w_n^k\right\|_{L^2}^2\right\}=0, \tag{6.171}$$

$$\lim_{n\to\infty}\left\{\mathcal{S}_\omega(\psi_n(0))-\sum_{j=1}^{J}\mathcal{S}_\omega\left(e^{-it_n^j\Delta}\widetilde{u}^j\right)-\mathcal{S}_\omega(w_n^J)\right\}=0, \tag{6.172}$$

$$\lim_{n\to\infty}\left\{\mathcal{I}_\omega(\psi_n(0))-\sum_{j=1}^{J}\mathcal{I}_\omega\left(e^{-it_n^j\Delta}\widetilde{u}^j\right)-\mathcal{I}_\omega(w_n^J)\right\}=0. \tag{6.173}$$

We shall show that $J=1$. Note that since $\mathcal{K}(\psi_n(0))>0$, we have $\mathcal{I}_\omega(\psi_n(0))<\mathcal{S}_\omega(\psi_n(0))$. It follows from (6.173) together with (6.43) that

$$\begin{aligned}&\mathcal{I}_\omega\left(e^{-it_n^j\Delta}\widetilde{u}^j\right),\quad \mathcal{I}_\omega(w_n^J)\\ &\le \mathcal{S}_\omega(\psi_n(0))<\frac{m_\omega+m_\omega^*}{2}\qquad \text{for any } 1\le j\le J \text{ and sufficiently large } n.\end{aligned} \tag{6.174}$$

Since $\widetilde{m}_\omega=m_\omega$ (see Proposition 1.1), we see from the the definition of $\widetilde{m}_\omega$ that

$$\mathcal{K}\left(e^{-it_n^j\Delta}\widetilde{u}^j\right)>0,\quad \mathcal{K}(w_n^J)>0\qquad \text{for any } 1\le j\le J \text{ and sufficiently large } n. \tag{6.175}$$

Moreover, we have by Lemma 6.1 together with (6.175) that

$$\liminf_{n\to\infty}\mathcal{S}_\omega\left(e^{-it_n^j\Delta}\widetilde{u}^j\right)>0\quad \text{for any } 1\le j\le J,\qquad \liminf_{n\to\infty}\mathcal{S}_\omega(w_n^J)\ge 0. \tag{6.176}$$

Now, suppose the contrary that $J\ge 2$. Then, it follows from (6.43), (6.172) and (6.176) that

$$\limsup_{n\to\infty}\mathcal{S}_\omega\left(e^{-it_n^j\Delta}\widetilde{u}^j\right)<m_\omega^*\qquad \text{for any } 1\le j\le J, \tag{6.177}$$

which together with (6.67) and the action-conservation law yields that

$$\mathcal{S}_\omega\left(\widetilde{\psi}^j(t)\right)<m_\omega^*,\qquad \text{for any } 1\le j\le J \text{ and } t\in I^j. \tag{6.178}$$

Since $\widetilde{\psi}^j$ is a solution to (NLS), it follows from the definition of m_ω^* that

$$\left\|\widetilde{\psi}^j\right\|_{W_{p_1}(I^j)\cap W(I^j)}<\infty\qquad \text{for any } 1\le j\le J. \tag{6.179}$$

This contradicts (6.167). Thus, we have $J=1$.

Since $\|\widetilde{\psi}^1\|_{W_{p_1}(I^1)\cap W(I^1)}=\infty$, we have

$$\mathcal{S}_\omega\left(\widetilde{\psi}^1(t)\right)\ge m_\omega^*\qquad \text{for any } t\in I^1. \tag{6.180}$$

On the other hand, we see from a proof similar to the one of (6.178) that

$$\mathcal{S}_\omega\left(\widetilde{\psi}^1(t)\right)\le m_\omega^*\qquad \text{for any } t\in I^1. \tag{6.181}$$

Combining (6.180) and (6.181), we obtain

$$\mathcal{S}_\omega\left(\widetilde{\psi}^1(t)\right) = m_\omega^* \qquad \text{for any } t \in I^1. \tag{6.182}$$

Since we have by (6.67) that

$$\mathcal{S}_\omega\left(\widetilde{\psi}^1(t)\right) = \mathcal{S}_\omega\left(e^{-it_n^1\Delta}\widetilde{u}^1\right), \tag{6.183}$$

(6.172) together with (6.43) and (6.182) shows

$$\lim_{n\to\infty} \mathcal{S}_\omega(w_n^1) = 0. \tag{6.184}$$

Hence, Lemma 6.1 together with (6.175) with $J = 1$ and (6.184) shows

$$\lim_{n\to\infty} \left\|w_n^1\right\|_{H^1} = 0. \tag{6.185}$$

We see from (6.52) together with (6.185) that

$$\lim_{n\to\infty} \left\|\psi_n(0) - e^{-it_n^1\Delta}\widetilde{u}^1(\cdot - x_n^1)\right\|_{H^1} = 0. \tag{6.186}$$

Now, we shall show that $I^1 = \mathbb{R}$. Suppose the contrary that $T^1 := \sup I^1 < \infty$. Let $\{t_n\}$ be a sequence in I^1 such that $\lim\limits_{n\to\infty} t_n \uparrow T_{\max}^1$, and put $\widetilde{\psi}_n(t) := \widetilde{\psi}^1(t+t_n)$ and $\widetilde{I}_n := I^1 - \tau_n^1$. We easily verify that the sequence $\{\widetilde{\psi}_n\}$ satisfies that

$$\widetilde{\psi}_n(t) \in A_{\omega,+} \qquad \text{for any } t \in \widetilde{I}_n, \tag{6.187}$$

$$\lim_{n\to\infty} \mathcal{S}_\omega(\widetilde{\psi}_n) = m_\omega^*, \tag{6.188}$$

$$\left\|\widetilde{\psi}_n\right\|_{W_{p_1}(\widetilde{I}_n)\cap W(\widetilde{I}_n)} = \infty. \tag{6.189}$$

Then, we can apply the above argument to this sequence, and find as well as (6.186) that there exists a non-trivial function $v \in H^1(\mathbb{R}^d)$, a sequence $\{\tau_n\}$ with $\tau_\infty := \lim\limits_{n\to\infty} \tau_n \in \mathbb{R} \cup \{\pm\infty\}$, and a sequence $\{\xi_n\}$, such that

$$\lim_{n\to\infty} \left\|\widetilde{\psi}^1(t_n) - e^{-i\tau_n\Delta}v(\cdot - \xi_n)\right\|_{H^1} = \lim_{n\to\infty} \left\|\widetilde{\psi}_n(0) - e^{-i\tau_n\Delta}v(\cdot - \xi_n)\right\|_{H^1} = 0. \tag{6.190}$$

This together with the Strichartz estimate also yields that

$$\lim_{n\to\infty} \left\|e^{it\Delta}\widetilde{\psi}^1(t_n) - e^{i(t-\tau_n)\Delta}v(\cdot - \xi_n)\right\|_{V_{p_1}(\mathbb{R})} = 0. \tag{6.191}$$

When $\tau_\infty = \pm\infty$, it follows from the decay estimate for the free solution that, for any compact interval I, we have

$$\lim_{n\to\infty} \left\|e^{i(t-\tau_n)\Delta}v\right\|_{V_{p_1}(I)} = 0, \tag{6.192}$$

which, with the help of (6.191), also yields that

$$\lim_{n\to\infty} \left\|e^{it\Delta}\widetilde{\psi}^1(t_n)\right\|_{V_{p_1}(I)} = 0. \tag{6.193}$$

On the other hand, when $\tau_\infty \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \left\| e^{it\Delta}\widetilde{\psi}^1(t_n) \right\|_{V_{p_1(I)}} = \left\| e^{i(t-\tau_\infty)\Delta} v \right\|_{V_{p_1(I)}} \ll 1 \quad \text{for any interval } I \text{ with } |I| \ll 1. \quad (6.194)$$

Then, Theorem 4.1 together with (6.193) and (6.194) implies that $\widetilde{\psi}^1$ exists beyond T^1, which is a contradiction. Thus, $\sup I^1 = +\infty$. Similarly, we have $\inf I^1 = -\infty$; Hence, $I^1 = \mathbb{R}$.

Put $\Psi := \widetilde{\psi}^1$. Then, this Ψ is what we want. □

6.3 Completion of the proof of Theorem 1.2

Since our critical element Ψ belongs to $C(\mathbb{R}, H^1(\mathbb{R}^d))$ (see Proposition 6.11), we can derive a contradiction in the same way as the energy subcritical case (see [3, 10]). Thus, we have $m_\omega^* = m_\omega$, which together with Theorem 4.1 (v) completes the proof of Theorem 1.2.

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